

# ASYMPTOTICS OF TOEPLITZ OPERATORS AND APPLICATIONS IN TQFT

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ABSTRACT. In this paper we provide a review of asymptotic results of Toeplitz operators and their applications in TQFT. To do this we review the differential geometric construction of the Hitchin connection on a prequantizable compact symplectic manifold. We use asymptotic results relating the Hitchin connection and Toeplitz operators, to, in the special case of the moduli space of flat  $SU(n)$ -connections on a surface, prove asymptotic faithfulness of the  $SU(n)$  quantum representations of the mapping class group. We then go on to review formal Hitchin connections and formal trivializations of these. We discuss how these fit together to produce a Berezin–Toeplitz star product, which is independent of the complex structure. Finally we give explicit examples of all the above objects in the case of the abelian moduli space. We furthermore discuss an approach to curve operators in the TQFT associated to abelian Chern–Simons theory.

## 1. INTRODUCTION

Witten constructed, via path integral techniques, a quantization of Chern–Simons theory in  $2 + 1$  dimensions, and he argued in [Wi] that this produced a TQFT, indexed by a compact simple Lie group and an integer level  $k$ . For the group  $SU(n)$  and level  $k$ , let us denote this TQFT by  $Z_k^{(n)}$ . Combinatorially, this theory was first constructed by Reshetikhin and Turaev, using representation theory of  $U_q(\mathfrak{sl}(n, \mathbb{C}))$  at  $q = e^{(2\pi i)/(k+n)}$ , in [RT1] and [RT2]. Subsequently, the TQFT’s  $Z_k^{(n)}$  were constructed using skein theory by Blanchet, Habegger, Masbaum and Vogel in [BHMV1], [BHMV2] and [B1].

The two-dimensional part of the TQFT  $Z_k^{(n)}$  is a modular functor with a certain label set. For this TQFT, the label set  $\Lambda_k^{(n)}$  is a finite subset (depending on  $k$ ) of the set of finite dimensional irreducible representations of  $SU(n)$ . We use the usual labeling of irreducible representations by Young diagrams, so in particular  $\square \in \Lambda_k^{(n)}$  is the defining representation of  $SU(n)$ . Let further  $\lambda_0^{(d)} \in \Lambda_k^{(n)}$  be the Young diagram consisting of  $d$  columns of length  $k$ . The label set is also equipped with an involution, which is simply induced by taking the dual representation. The trivial representation is a special element in the label set which is clearly preserved by the involution.

$$Z_k^{(n)} : \left\{ \begin{array}{l} \text{Category of (ex-} \\ \text{tended) closed} \\ \text{oriented surfaces} \\ \text{with } \Lambda_k^{(n)}\text{-labeled} \\ \text{marked points with} \\ \text{projective tangent} \\ \text{vectors} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{Category of finite} \\ \text{dimensional vector} \\ \text{spaces over } \mathbb{C} \end{array} \right\}$$

The three-dimensional part of  $Z_k^{(n)}$  is an association of a vector,

$$Z_k^{(n)}(M, L, \lambda) \in Z_k^{(n)}(\partial M, \partial L, \partial \lambda),$$

to any compact, oriented, framed 3-manifold  $M$  together with an oriented, framed link  $(L, \partial L) \subseteq (M, \partial M)$  and a  $\Lambda_k^{(n)}$ -labeling  $\lambda : \pi_0(L) \rightarrow \Lambda_k^{(n)}$ .

This association has to satisfy the Atiyah-Segal-Witten TQFT axioms (see e.g. [At], [Se] and [Wi]). For a more comprehensive presentation of the axioms, see Turaev's book [T].

The geometric construction of these TQFTs was proposed by Witten in [Wi] where he derived, via the Hamiltonian approach to quantum Chern-Simons theory, that the geometric quantization of the moduli spaces of flat connections should give the two-dimensional part of the theory. Further, he proposed an alternative construction of the two-dimensional part of the theory via WZW-conformal field theory. This theory has been studied intensively. In particular, the work of Tsuchiya, Ueno and Yamada in [TUY] provided the major geometric constructions and results needed. In [BK], their results were used to show that the category of integrable highest weight modules of level  $k$  for the affine Lie algebra associated to any simple Lie algebra is a modular tensor category. Further, in [BK], this result is combined with the work of Kazhdan and Lusztig [KL] and the work of Finkelberg [Fi] to argue that this category is isomorphic to the modular tensor category associated to the corresponding quantum group, from which Reshetikhin and Turaev constructed their TQFT. Unfortunately, these results do not allow one to conclude the validity of the geometric constructions of the two-dimensional part of the TQFT proposed by Witten. However, in joint work with Ueno, [AU1], [AU2], [AU3] and [AU4], the first author have given a proof, based mainly on the results of [TUY], that the TUY-construction of the WZW-conformal field theory, after twist by a fractional power of an abelian theory, satisfies all the axioms of a modular functor. Furthermore, we have proved that the full 2 + 1-dimensional TQFT resulting from this is isomorphic to the aforementioned one, constructed by BHMV via skein theory. Combining this with the theorem of Laszlo [La1], which identifies (projectively) the representations of the mapping class groups obtained from the geometric quantization of the moduli space of flat connections with the ones obtained from the TUY-constructions, one gets a proof of the validity of the construction proposed by Witten in [Wi].

Another part of this TQFT is the quantum  $SU(n)$  representations of the mapping class groups. Namely, if  $\Sigma$  is a closed oriented surfaces of genus  $g$ ,  $\Gamma$  is the mapping class group of  $\Sigma$ , and  $p$  is a point on  $\Sigma$ , then the modular functor induces a representation

$$(1) \quad Z_k^{(n,d)} : \Gamma \rightarrow \mathbb{P} \text{Aut}(Z_k^{(n)}(\Sigma, p, \lambda_0^{(d)})).$$

For a general label of  $p$ , we would need to choose a projective tangent vector  $v_p \in T_p \Sigma / \mathbb{R}_+$ , and we would get a representation of the mapping class group of  $(\Sigma, p, v_p)$ . But for the special labels  $\lambda_0^{(d)}$ , the dependence on  $v_p$  is trivial and in fact we get a representation of  $\Gamma$ .

Let us now briefly recall the geometric construction of the representations  $Z_k^{(n,d)}$  of the mapping class group, as proposed by Witten, using geometric quantization of moduli spaces.

We assume from now on that the genus of the closed oriented surface  $\Sigma$  is at least two. Let  $M$  be the moduli space of flat  $SU(n)$  connections on  $\Sigma - p$  with holonomy around  $p$  equal to  $\exp(2\pi i d/n) \text{Id} \in SU(n)$ . When  $(n, d)$  are coprime, the moduli space is smooth. In all cases, the smooth part of the moduli space has a natural symplectic structure  $\omega$ . There is a natural smooth symplectic action of the mapping class group  $\Gamma$  of  $\Sigma$  on  $M$ . Moreover, there is a unique prequantum line bundle  $(\mathcal{L}, \nabla, (\cdot, \cdot))$  over  $(M, \omega)$ . The Teichmüller space  $\mathcal{T}$  of complex structures on  $\Sigma$  naturally, and  $\Gamma$ -equivariantly, parametrizes Kähler structures on  $(M, \omega)$ . For  $\sigma \in \mathcal{T}$ , we denote by  $M_\sigma$  the manifold  $(M, \omega)$  with its corresponding Kähler structure. The complex structure on  $M_\sigma$  and the connection  $\nabla$  in  $\mathcal{L}$  induce the structure of a holomorphic line bundle on  $\mathcal{L}$ . This holomorphic line bundle is simply the determinant line bundle over the moduli space, and it is an ample generator of the Picard group [DN].

By applying geometric quantization to the moduli space  $M$ , one gets, for any positive integer  $k$ , a certain finite rank bundle over Teichmüller space  $\mathcal{T}$  which we will call the *Verlinde bundle*  $\mathcal{V}^{(k)}$  at level  $k$ . The fiber of this bundle over a point  $\sigma \in \mathcal{T}$  is  $\mathcal{V}_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}^k)$ . We observe that there is a natural Hermitian structure  $\langle \cdot, \cdot \rangle$  on  $H^0(M_\sigma, \mathcal{L}^k)$  by restricting the  $L_2$ -inner product on global  $L_2$  sections of  $\mathcal{L}^k$  to  $H^0(M_\sigma, \mathcal{L}^k)$ .

The main result pertaining to this bundle is:

**Theorem 1** (Axelrod, Della Pietra and Witten; Hitchin). *The projectivization of the bundle  $\mathcal{V}^{(k)}$  supports a natural flat  $\Gamma$ -invariant connection  $\hat{\nabla}$ .*

This is a result proved independently by Axelrod, Della Pietra and Witten [ADW] and by Hitchin [H]. In section 2, we review our differential geometric construction of the connection  $\hat{\nabla}$  in the general setting discussed in [A6]. We obtain as a corollary that the connection constructed by Axelrod, Della Pietra and Witten projectively agrees with Hitchin's.

Because of the existence of this connection, the 2-dimensional part of the modular functor  $Z_k^{(n)}$  is the vector space  $\mathbb{P}(V^{(k)})$  of covariant constant sections of  $\mathbb{P}(\mathcal{V}^{(k)})$  over Teichmüller space  $\mathcal{T}$ .

**Definition 1.** *We denote by  $Z_k^{(n,d)}$  the representation,*

$$Z_k^{(n,d)} : \Gamma \rightarrow \text{Aut}(\mathbb{P}(V^{(k)})),$$

*obtained from the action of the mapping class group on the covariant constant sections of  $\mathbb{P}(\mathcal{V}^{(k)})$  over  $\mathcal{T}$ .*

The projectively flat connection  $\hat{\nabla}$  induces a *flat* connection  $\hat{\nabla}^e$  in  $\text{End}(\mathcal{V}^{(k)})$ . This flat connection can be used to show asymptotically flatness of the quantum representations  $Z_k^{(n,d)}$ ,

**Theorem 2** (Andersen [A3]). *Assume that  $g \geq 2$ ,  $n$  and  $d$  are coprime or that  $(n, d) = (2, 0)$  when  $g = 2$ . Then, we have that*

$$\bigcap_{k=1}^{\infty} \ker(Z_k^{(n,d)}) = \begin{cases} \{1, H\} & g = 2, n = 2 \text{ and } d = 0 \\ \{1\} & \text{otherwise,} \end{cases}$$

where  $H$  is the hyperelliptic involution.

In Section 4 we discuss the proof of this Theorem, and how it relies on the asymptotics of Toeplitz operators  $T_f^{(k)}$  associated a smooth function  $f$  on  $M$ . For each  $f \in C^\infty(M)$  and each point  $\sigma \in \mathcal{T}$  we have the Toeplitz operator,

$$T_{f,\sigma}^{(k)} : H^0(M_\sigma, \mathcal{L}_\sigma^k) \rightarrow H^0(M_\sigma, \mathcal{L}_\sigma^k),$$

which is given by

$$T_{f,\sigma}^{(k)} s = \pi_\sigma^{(k)}(fs)$$

for all  $s \in H^0(M_\sigma, \mathcal{L}_\sigma^k)$ . Here  $\pi_\sigma^{(k)}$  is the orthogonal projection onto  $H^0(M_\sigma, \mathcal{L}_\sigma^k)$  induced from the  $L_2$ -inner product on  $C^\infty(M, \mathcal{L}^k)$ . We get a smooth section of  $\text{End}(\mathcal{V}^{(k)})$ ,

$$T_f^{(k)} \in C^\infty(\mathcal{T}, \text{End}(\mathcal{V}^{(k)})),$$

by letting  $T_f^{(k)}(\sigma) = T_{f,\sigma}^{(k)}$ . See Section 3 for a discussion of the Toeplitz operators and their connection to deformation quantization. The sections  $T_f^{(k)}$  of  $\text{End} \mathcal{V}^{(k)}$  over  $\mathcal{T}$  are not covariant constant with respect to  $\hat{\nabla}^e$ . However, they are asymptotically as  $k$  goes to infinity. This is made precise when we discuss the formal Hitchin Connection below.

The existence of a connection as above is not a unique thing for the moduli spaces, the construction can be generalized to a general compact prequantizable symplectic manifold  $(M, \omega)$  with prequantum line bundle  $(\mathcal{L}, (\cdot, \cdot), \nabla)$ . We assume that  $\mathcal{T}$  is a complex manifold which holomorphically and rigidly (see Definition 6) parameterizes Kähler structures on  $(M, \omega)$ . Then, the following theorem, proved in [A6], establishes the existence of the Hitchin connection (see Definition 7) under a mild cohomological condition.

**Theorem 3** (Andersen). *Suppose that  $I$  is a rigid family of Kähler structures on the compact, prequantizable symplectic manifold  $(M, \omega)$  which satisfies that there exists an  $n \in \mathbb{Z}$  such that the first Chern class of  $(M, \omega)$  is  $n[\frac{\omega}{2\pi}] \in H^2(M, \mathbb{Z})$  and  $H^1(M, \mathbb{R}) = 0$ . Then, the Hitchin connection  $\hat{\nabla}$  in the trivial bundle  $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$  preserves the subbundle  $H^{(k)}$  with fibers  $H^0(M_\sigma, \mathcal{L}^k)$ . It is given by*

$$\hat{\nabla}_V = \hat{\nabla}_V^t + \frac{1}{4k + 2n} \{ \Delta_{G(V)} + 2\nabla_{G(V) \cdot dF} + 4kV'[F] \},$$

where  $\hat{\nabla}^t$  is the trivial connection in  $\mathcal{H}^{(k)}$ , and  $V$  is any smooth vector field on  $\mathcal{T}$ .

This result is discussed in much greater detail in Section 2, where all ingredients are introduced.

In Section 5, we study the formal Hitchin connection which was introduced in [A6]. Let  $\mathcal{D}(M)$  be the space of smooth differential operators on  $M$  acting on smooth functions on  $M$ . Let  $\mathbb{C}_h$  be the trivial  $C_h^\infty(M)$ -bundle over  $\mathcal{T}$ , where  $C_h^\infty(M)$  is formal power series with coefficients in  $C^\infty(M)$ .

**Definition 2.** A formal connection  $D$  is a connection in  $\mathbb{C}_h$  over  $\mathcal{T}$  of the form

$$D_V f = V[f] + \tilde{D}(V)(f),$$

where  $\tilde{D}$  is a smooth one-form on  $\mathcal{T}$  with values in  $\mathcal{D}_h(M) = \mathcal{D}(M)[[h]]$ ,  $f$  is any smooth section of  $\mathbb{C}_h$ ,  $V$  is any smooth vector field on  $\mathcal{T}$  and  $V[f]$  is the derivative of  $f$  in the direction of  $V$ .

Thus, a formal connection is given by a formal series of differential operators

$$\tilde{D}(V) = \sum_{l=0}^{\infty} \tilde{D}^{(l)}(V) h^l.$$

From Hitchin's connection in  $H^{(k)}$ , we get an induced connection  $\hat{\nabla}^e$  in the endomorphism bundle  $\text{End}(H^{(k)})$ . As previously mentioned, the Toeplitz operators are not covariant constant sections with respect to  $\hat{\nabla}^e$ , but asymptotically in  $k$  they are. This follows from the properties of the formal Hitchin connection, which is the formal connection  $D$  defined through the following theorem (proved in [A6]).

**Theorem 4.** (Andersen) *There is a unique formal connection  $D$  which satisfies that*

$$(2) \quad \hat{\nabla}_V^e T_f^{(k)} \sim T_{(D_V f)(1/(k+n/2))}^{(k)}$$

for all smooth section  $f$  of  $\mathbb{C}_h$  and all smooth vector fields  $V$  on  $\mathcal{T}$ . Moreover,

$$\tilde{D} = 0 \mod h.$$

Here  $\sim$  means the following: For all  $L \in \mathbb{Z}_+$  we have that

$$\left\| \hat{\nabla}_V^e T_f^{(k)} - \left( T_{V[f]}^{(k)} + \sum_{l=1}^L T_{\tilde{D}^{(l)} f}^{(k)} \frac{1}{(k+n/2)^l} \right) \right\| = O(k^{-(L+1)}),$$

uniformly over compact subsets of  $\mathcal{T}$ , for all smooth maps  $f : \mathcal{T} \rightarrow C^\infty(M)$ .

Now fix an  $f \in C^\infty(M)$ , which does not depend on  $\sigma \in \mathcal{T}$ , and notice how the fact that  $\tilde{D} = 0 \mod h$  implies that

$$\left\| \hat{\nabla}_V^e T_f^{(k)} \right\| = O(k^{-1}).$$

This expresses the fact that the Toeplitz operators are asymptotically flat with respect to the Hitchin connection.

We define a mapping class group equivariant formal trivialization of  $D$  as follows.

**Definition 3.** A formal trivialization of a formal connection  $D$  is a smooth map  $P : \mathcal{T} \rightarrow \mathcal{D}_h(M)$  which modulo  $h$  is the identity, for all  $\sigma \in \mathcal{T}$ , and which satisfies

$$D_V(P(f)) = 0,$$

for all vector fields  $V$  on  $\mathcal{T}$  and all  $f \in C_h^\infty(M)$ . Such a formal trivialization is mapping class group equivariant if  $P(\phi(\sigma)) = \phi^* P(\sigma)$  for all  $\sigma \in \mathcal{T}$  and  $\phi \in \Gamma$ .

Since the only mapping class group invariant functions on the moduli space are the constant ones (see [Go1]), we see that in the case where  $M$  is the moduli space, such a  $P$ , if it exists, must be unique up to multiplication by a formal constant, i.e. an element of  $\mathbb{C}_h = \mathbb{C}[[h]]$ .

Clearly if  $D$  is not flat, such a formal trivialization cannot exist even locally on  $\mathcal{T}$ . However, if  $D$  is flat and its zero-order term is just given by the trivial connection in  $C_h$ , then a local formal trivialization exists, as proved in [A6].

Furthermore, it is proved in [A6] that flatness of the formal Hitchin connection is implied by projective flatness of the Hitchin connection. As was proved by Hitchin in [H], and stated above in Theorem 1, this is the case when  $M$  is the moduli space.

In Section 5 we discuss how this formal trivialization of a formal connection give a way of defining a star product from the Berezin–Topelitz star product, which turn out not to depend on the complex structure  $\sigma$ . In Section 5 we furthermore discuss the lower order terms of formal trivialization and the star product.

In Section 6 we consider the all of the above objects in the case where the manifold  $M$  is a principal polarized abelian variety. We furthermore discuss abelian Chern–Simons theory and the moduli space of  $U(1)$ -connections on a closed surface  $\Sigma$ . We find a flat Hitchin connection on the  $U(1)$ -moduli space  $M$  and find a formal trivialization  $P$  of the formal Hitchin connection. With this formal trivialization we define the curve operators to a cylinder  $\Sigma \times [0, 1]$  with a link  $\gamma$  inside, to be the Toeplitz operator associated to the corresponding holonomy function  $h_\gamma$  on  $M$ ,  $Z^{(k)} = T_{h_\gamma}^{(k)}$ . With this definition of a curve operator we show that

$$\langle h_{\gamma_1}, h_{\gamma_2} \rangle = \lim_{k \rightarrow \infty} \langle Z^{(k)}(\Sigma, \gamma_1), Z^{(k)}(\Sigma, \gamma_2) \rangle$$

and as required by the TQFT axioms that

$$Z^{(k)}(\Sigma \times S^1) = \dim(Z^{(k)}(\Sigma)).$$

## 2. THE HITCHIN CONNECTION

In this section, we review our construction of the Hitchin connection using the global differential geometric setting of [A6]. This approach is close in spirit to Axelrod, Della Pietra and Witten’s in [ADW], however we do not use any infinite dimensional gauge theory. In fact, the setting is more general than the gauge theory setting in which Hitchin in [H] constructed his original connection. But when applied to the gauge theory situation, we get the corollary that Hitchin’s connection agrees with Axelrod, Della Pietra and Witten’s.

Hence, we start in the general setting and let  $(M, \omega)$  be any compact symplectic manifold.

**Definition 4.** A prequantum line bundle  $(\mathcal{L}, (\cdot, \cdot), \nabla)$  over the symplectic manifold  $(M, \omega)$  consist of a complex line bundle  $\mathcal{L}$  with a Hermitian structure  $(\cdot, \cdot)$  and a compatible connection  $\nabla$  whose curvature is

$$F_\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} = -i\omega(X, Y).$$

We say that the symplectic manifold  $(M, \omega)$  is prequantizable if there exist a prequantum line bundle over it.

Recall that the condition for the existence of a prequantum line bundle is that  $[\frac{\omega}{2\pi}] \in \text{Im}(H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{R}))$ . Furthermore, the inequivalent choices of prequantum line bundles (if they exist) are parametrized by  $H^1(M, U(1))$  (see e.g. [Wo]).

We shall assume that  $(M, \omega)$  is prequantizable and fix a prequantum line bundle  $(\mathcal{L}, (\cdot, \cdot), \nabla)$ .

Before dwelling into the details we discuss general facts about families of Kähler structures on a symplectic manifold.

**Families of Kähler structures.** From now on we assume  $\mathcal{T}$  is a smooth manifold. Later we impose extra structure.

A *family of Kähler structures* on a symplectic manifold  $(M, \omega)$  parametrized by  $\mathcal{T}$  is a map

$$I : \mathcal{T} \rightarrow C^\infty(M, \text{End } TM),$$

that to each element  $\sigma \in \mathcal{T}$  associates an integrable and compatible almost complex structure.  $I$  is said to be smooth if  $I$  defines a smooth section of  $\pi_M^* \text{End}(TM) \rightarrow \mathcal{T} \times M$ .

For each point  $\sigma \in \mathcal{T}$  we define  $M_\sigma$  to be  $M$  with the Kähler structure defined by  $\omega$  and  $I_\sigma := I(\sigma)$ , and the Kähler metric is denoted by  $g_\sigma$ .

Every  $I_\sigma$  is an almost complex structure and hence induce a splitting of the complexified tangent bundle  $TM_\mathbb{C}$ , denoted by  $TM_\mathbb{C} = T_\sigma \oplus \bar{T}_\sigma$ , and the projection to each factor is given by

$$\pi_\sigma^{1,0} = \frac{1}{2}(Id - iI_\sigma) \quad \text{and} \quad \pi_\sigma^{0,1} = \frac{1}{2}(Id + iI_\sigma).$$

If  $I_\sigma^2 = -Id$  is differentiated along a vector field  $V$  on  $\mathcal{T}$ , we get

$$V[I]_\sigma I_\sigma + I_\sigma V[I]_\sigma = 0,$$

and hence  $V[I]_\sigma$  changes types on  $M_\sigma$ . Then for each  $\sigma$ ,  $V[I]_\sigma$  give an element of

$$C^\infty(M, ((\bar{T}_\sigma)^* \otimes T_\sigma) \oplus ((T_\sigma)^* \otimes \bar{T}_\sigma)),$$

and we have a splitting  $V[I]_\sigma = V[I]_\sigma' + V[I]_\sigma''$  where

$$V[I]_\sigma' \in C^\infty(M, (\bar{T}_\sigma)^* \otimes T_\sigma) \quad \text{and} \quad V[I]_\sigma'' \in C^\infty(M, (T_\sigma)^* \otimes \bar{T}_\sigma).$$

This splitting of  $V[I]$  happens for every vector field on  $\mathcal{T}$  and actually induce an almost complex structure on  $\mathcal{T}$ .

Since  $V[I]_\sigma$  is a smooth section of  $TM_\mathbb{C} \otimes T^*M_\mathbb{C}$  and the symplectic structure is a smooth section of  $T^*M_\mathbb{C} \otimes T^*M_\mathbb{C}$  we can define a bivector field  $\tilde{G}(V)$  by contraction with the symplectic form

$$\tilde{G}(V) \cdot \omega = V[I].$$

$\tilde{G}(V)$  is unique since  $\omega$  is non-degenerate. By definition of the Kähler metric,  $g$  is the contraction of  $\omega$  and  $I$ ,  $g = \omega \cdot I$ . We use the  $\cdot$ -notation for contraction in the following way

$$g(X, Y) = (\omega \cdot I)(X, Y) = \omega(X, IY) \quad \text{and} \quad g(X, Y) = -(I \cdot \omega)(X, Y) = -\omega(IX, Y).$$

Since  $\omega$  is independent of  $\sigma$  taking the derivative of this identity in the direction of a vector field  $V$  on  $\mathcal{T}$  we obtain

$$V[g] = \omega \cdot V[I] = \omega \cdot \tilde{G}(V) \cdot \omega.$$

Since  $g$  is symmetric so is  $V[g]$ , and with  $\omega$  being anti-symmetric  $\tilde{G}(V)$  is symmetric. We furthermore have that  $\omega$  is of type  $(1, 1)$  when regarded as a Kähler form on  $M_\sigma$ , using this and the fact that  $V[I]_\sigma$  changes types on  $M_\sigma$  we get that  $\tilde{G}(V)$  splits as  $\tilde{G}(V) = G(V) + \bar{G}(V)$ , where

$$G(V) \in C^\infty(M, S^2(T')) \quad \text{and} \quad \bar{G}(V) \in C^\infty(M, S^2(\bar{T})).$$

In the above we have suppressed the dependence on  $\sigma$ , since this is valid for any  $\sigma$ .

**Holomorphic families of Kähler structures.** Let us now assume that  $\mathcal{T}$  furthermore is a complex manifold. We can then ask  $I : \mathcal{T} \rightarrow C^\infty(M, \text{End}(TM))$  to be holomorphic. By using the splitting of  $V[I]$  we make the following definition.

**Definition 5.** *Let  $\mathcal{T}$  be a complex manifold and  $I$  a smooth family of complex structures on  $M$  parametrized by  $\mathcal{T}$ . Then  $I$  is holomorphic if*

$$V'[I] = V[I]' \quad \text{and} \quad V''[I] = V[I]''$$

for all vector fields  $V$  on  $\mathcal{T}$ .

Assume  $J$  is an integrable almost complex structure on  $\mathcal{T}$  induced by the complex structure on  $\mathcal{T}$ .  $J$  induces an almost complex structure,  $\hat{I}$  on  $\mathcal{T} \times M$  by

$$\hat{I}(V \oplus X) = JV \oplus I_\sigma X,$$

where  $V + X \in T_{(\sigma, p)}(\mathcal{T} \times M)$ . In [AGL] a simple calculation shows that the Nijenhuis tensor on  $\mathcal{T} \times M$  vanish exactly when  $\pi^{0,1}V'[I]X = 0$  and  $\pi^{1,0}V''[I]X = 0$ , which by the Newlander–Nirenberg theorem shows that  $\hat{I}$  is integrable if and only if  $I$  is holomorphic, hence the name.

Remark that for a holomorphic family of Kähler structures on  $(M, \omega)$  we have

$$\tilde{G}(V') \cdot \omega = V'[I] = V[I]' = G(V) \cdot \omega,$$

which implies  $\tilde{G}(V') = G(V)$ . We can in the same way show that  $\bar{G}(V) = \tilde{G}(V'')$ .

**Rigid families of Kähler structures.** In constructing an explicit formula for the Hitchin connection we need the following rather restrictive assumption on our family of Kähler structures.

**Definition 6.** *A family of Kähler structures  $I$  on  $M$  is called rigid if*

$$\nabla_{X''}G(V) = 0$$

for all vector fields  $V$  on  $\mathcal{T}$  and  $X$  on  $M_\sigma$ .

Equivalently we could give the above equation in terms of the induced  $\bar{\partial}_\sigma$ -operator on  $M_\sigma$ ,

$$\bar{\partial}_\sigma(G(V)_\sigma) = 0,$$

for all  $\sigma \in \mathcal{T}$  and all vector fields  $V$  on  $\mathcal{T}$ .

There are several examples of rigid families of Kähler structures, see e.g. [AGL]. It should be remarked that this condition is also built into the arguments of [H].

**The Hitchin connection.** Now all tools are defined and we can construct the Hitchin connection. In Theorem 5 we need  $M$  to be compact, so let us assume this. Recall the quantum spaces

$$H_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}_\sigma^k) \{s \in C^\infty(M, \mathcal{L}^k) \mid \nabla_\sigma^{0,1}s = 0\},$$

where  $\nabla_\sigma^{0,1} = \frac{1}{2}(Id + iI_\sigma)\nabla$ .

It is not clear that these spaces form a vector bundle over  $\mathcal{T}$ . But by constructing a bundle, where these sit as subspaces of each of the fibers, and a connection in this bundle preserving  $H_\sigma^{(k)}$ ,  $H^{(k)}$  will be a subbundle over  $\mathcal{T}$ .

Define the trivial bundle  $\mathcal{H}^{(k)} = \mathcal{T} \times C^\infty(M, \mathcal{L}^k)$  of infinite rank. The finite dimensional subspaces  $H_\sigma^{(k)}$  sits inside each of the fibers. This bundle has of course the trivial connection  $\nabla^t$ , but we seek a connection preserving  $H_\sigma^{(k)}$ .



**Definition 7.** A Hitchin connection is a connection  $\hat{\nabla}$  in  $\mathcal{H}^{(k)}$ , which preserves the subspaces  $H_\sigma^{(k)}$ , and is of the form

$$\hat{\nabla} = \nabla^t + u,$$

where  $u \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}^k))$  is a 1-form on  $\mathcal{T}$  with values in differential operators acting on sections of  $\mathcal{L}^k$ .

By analyzing the condition  $\nabla_\sigma^{0,1} \hat{\nabla}_V s = 0$  for every vector field  $V$  on  $\mathcal{T}$ , we hope to find an explicit expression for  $u$ . If we express the above condition in terms of  $u$ ,  $u$  should satisfy

$$0 = \nabla_\sigma^{0,1} V[s] + \nabla_\sigma^{0,1} u(V)s,$$

and if we differentiate  $\nabla_\sigma^{0,1} s = 0$  along the a vector field  $V$  on  $\mathcal{T}$  we get

$$0 = V[\nabla_\sigma^{0,1} s] = V[\frac{1}{2}(Id + iI_\sigma)\nabla s] = \frac{i}{2}V[I_\sigma]\nabla s + \nabla_\sigma^{0,1} V[s].$$

If we combine the previous two equations we get the following

**Lemma 1.** The connection  $\hat{\nabla} = \nabla^t + u$  preserves  $H_\sigma^{(k)}$  for all  $\sigma \in \mathcal{T}$  if and only if  $u$  satisfy the equation

$$(3) \quad \nabla_\sigma^{0,1} u(V)s = \frac{i}{2}V[I]_\sigma \nabla_\sigma^{1,0} s$$

for all  $\sigma \in \mathcal{T}$  and all vector fields  $V$  on  $\mathcal{T}$ .

If the conclusion is true the collection of subspaces  $H_\sigma^{(k)} \subset C^\infty(M, \mathcal{L}^k)$  constitute a subbundle  $H^{(k)}$  of  $\mathcal{H}^{(k)}$ .

Let us now assume that  $\mathcal{T}$  is a complex manifold, and that the family  $I$  is holomorphic. First of all,  $\nabla_\sigma^{1,0} s$  is a section of  $(T_\sigma)^* \otimes \mathcal{L}^k$ , so it is constant in the  $\bar{T}_\sigma$ -direction, which is why  $V[I]_\sigma'' \nabla_\sigma^{1,0} s = 0$ , and by holomorphicity  $V''[I] = V[I]''$ , so  $V''[I]_\sigma \nabla_\sigma^{1,0} s = 0$ . Hence we can choose  $u(V'') = 0$ , and we therefore only need to focus on  $u$  in the  $V'$ -direction.

$u(V)$  should be a differential operator acting on sections of  $\mathcal{L}^k$ , and be related to  $I$ , so let us construct an operator from  $I$ .

Given a smooth symmetric bivector field  $B$  on  $M$  we define a differential operator on smooth sections of  $\mathcal{L}^k$  by

$$\Delta_B = \nabla_B^2 + \nabla_{\delta B},$$

where  $\delta B$  is the divergence of a symmetric bivector field

$$\delta_\sigma(B) = \text{Tr } \nabla_\sigma B.$$

$\nabla_B^2$  is defined by

$$\nabla_{X,Y}^2 = \nabla_X \nabla_Y s - \nabla_{\nabla_X Y} s,$$

which is tensorial in the vector fields  $X$  and  $Y$ . Thus we can evaluate it on a bivector field, and have thus defined  $\Delta_B$ .

Recall the bivector field  $G(V)$  defined by  $G(V) \cdot \omega = V'[I]$ . Using the above construction give a differential operator  $\Delta_{G(V)} : C^\infty(M, \mathcal{L}^k) \rightarrow C^\infty(M, \mathcal{L}^k)$ . Locally  $G(V) = \sum_j X_j \otimes Y_j$ , and

$$(4) \quad \Delta_{G(V)} = \nabla_{G(V)}^2 + \nabla_{\delta G(V)} = \sum_j \nabla_{X_j} \nabla_{Y_j} + \nabla_{\delta(X_j)Y_j},$$

since  $\delta(X_j \otimes Y_j) = \delta(X_j)Y_j + \nabla_{X_j}Y_j$ , where  $\delta(X)$  is the usual divergence of a vector field, which can be defined in many ways e.g. in terms of the Levi-Cevita connection on  $M_\sigma$  by  $\delta(X) = \text{Tr } \nabla_\sigma X$ .

The second order differential operator  $\Delta_{G(V)}$  is the cornerstone in the construction of  $u(V)$ . The idea in Andersen's construction is to calculate  $\nabla^{0,1}\Delta_{G(V)}s$  and find remainder terms, which cancel other terms such that  $\Delta_{G(V)}$  with these correction terms satisfy equation (3). When calculating  $\nabla^{0,1}\Delta_{G(V)}s$  the trace of the curvature of  $M_\sigma$  show up – that is the Ricci curvature  $\text{Ric}_\sigma$ . From Hodge decomposition  $\text{Ric}_\sigma = \text{Ric}_\sigma^H + 2i\partial_\sigma\bar{\partial}_\sigma F_\sigma$  where  $\text{Ric}_\sigma^H$  is harmonic and  $F_\sigma$  is the Ricci potential. As with the family of Kähler structures the Ricci potentials  $F_\sigma$  is a family of Ricci potentials parametrized by  $\mathcal{T}$  and can therefore be differentiated along a vector field on  $\mathcal{T}$ .

Define  $u \in \Omega^1(\mathcal{T}, \mathcal{D}(M, \mathcal{L}^k))$  by

$$(5) \quad u(V) = \frac{1}{4k+2n}(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 4kV'[F]).$$

**Theorem 5** (Andersen [A6]). *Let  $(M, \omega)$  be a compact prequantizable symplectic manifold with  $H^1(M, \mathbb{R}) = 0$  and first Chern class  $c_1(M, \omega) = n[\frac{\omega}{2\pi}]$ . Let  $I$  be a rigid holomorphic family of Kähler structures on  $M$  parametrized by a complex manifold  $\mathcal{T}$ . Then*

$$\hat{\nabla}_V = \nabla_V^t + \frac{1}{4k+2n}(\Delta_{G(V)} + 2\nabla_{G(V)\cdot dF} + 4kV'[F])$$

*is a Hitchin connection in the bundle  $H^{(k)}$  over  $\mathcal{T}$ .*

**Remark 1.** *The condition  $c_1(M, \omega) = n[\frac{\omega}{2\pi}] = nc_1(\mathcal{L})$  can be removed by switching to the metaplectic correction. Here we make the same construction but now a square root of the canonical bundle of  $(M, \omega)$  is tensored onto  $\mathcal{L}^k$ . Such a square root exists exactly if the second Stiefel–Whitney class is 0 – that is if  $M$  is spin, see [AGL] and also [Ch].*

**Remark 2.** *The condition  $H^1(M, \mathbb{R}) = 0$  is used to make the calculations in the proof easier, but there is no known examples of manifolds with  $H^1(M, \mathbb{R}) \neq 0$ , which satisfy the remaining conditions where the Hitchin connection cannot be built in this way. An example is the torus  $T^{2n}$  which we will study in much greater detail in Section 6.*

**Remark 3.** *By using Toeplitz operator theory, it can be shown that under some further assumptions on the family of Kähler structures the Hitchin connection is actually projectively flat. A proof of this can be found in [G].*

Suppose  $\Gamma$  is a group which acts by bundle automorphisms of  $\mathcal{L}$  over  $M$  preserving both the Hermitian structure and the connection in  $\mathcal{L}$ . Then there is an induced action of  $\Gamma$  on  $(M, \omega)$ . We will further assume that  $\Gamma$  acts on  $\mathcal{T}$  and that  $I$  is  $\Gamma$ -equivariant. In this case we immediately get the following invariance.

**Lemma 2.** *The natural induced action of  $\Gamma$  on  $\mathcal{H}^{(k)}$  preserves the subbundle  $H^{(k)}$  and the Hitchin connection.*

We are actually interested in the induced connection  $\hat{\nabla}^e$  in the endomorphism bundle  $\text{End}(H^{(k)})$ . Suppose  $\Phi$  is a section of  $\text{End}(H^{(k)})$ . Then for all sections  $s$  of  $H^{(k)}$  and all vector fields  $V$  on  $\mathcal{T}$ , we have that

$$(\hat{\nabla}_V^e \Phi)(s) = \hat{\nabla}_V \Phi(s) - \Phi(\hat{\nabla}_V(s)).$$

Assume now that we have extended  $\Phi$  to a section of  $\text{Hom}(\mathcal{H}^{(k)}, H^{(k)})$  over  $\mathcal{T}$ . Then

$$(6) \quad \hat{\nabla}_V^e \Phi = \hat{\nabla}_V^{e,t} \Phi + [\Phi, u(V)],$$

where  $\hat{\nabla}^{e,t}$  is the trivial connection in the trivial bundle  $\text{End}(\mathcal{H}^{(k)})$  over  $\mathcal{T}$ .

### 3. TOEPLITZ OPERATORS ON COMPACT KÄHLER MANIFOLDS

In this section we discuss the Toeplitz operators on compact Kähler manifolds  $(M, \omega)$  with Kähler structures parametrized by a smooth manifold  $\mathcal{T}$  and their asymptotics as the level  $k$  goes to infinity.

For each  $f \in C^\infty(M)$  we consider the differential operator  $M_f^{(k)} : C^\infty(M, \mathcal{L}^k) \rightarrow C^\infty(M, \mathcal{L}^k)$  given by

$$M_f^{(k)}(s) = fs$$

for all  $s \in H^0(M, \mathcal{L}^k)$ .

These operators act on  $C^\infty(M, \mathcal{L}^k)$  and therefore also on the trivial bundle  $\mathcal{H}^{(k)}$ , however they do not preserve the subbundle  $H^{(k)}$ . There is however a solution to this, which is given by the Hilbert space structure. Integrating the inner product of two sections of  $\mathcal{L}^k$  against the volume form associated to the symplectic form gives the pre-Hilbert space structure on  $C^\infty(M)$

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M (s_1, s_2) \omega^m.$$

This is not only a pre-Hilbert space structure on  $C^\infty(M, \mathcal{L}^k)$  but also on the trivial bundle  $\mathcal{H}^{(k)}$  which is of course compatible with the trivial connection in this bundle. This pre-Hilbert space structure induces a Hermitian structure  $\langle \cdot, \cdot \rangle$  on the finite rank subbundle  $H^{(k)}$  of  $\mathcal{H}^{(k)}$ . The Hermitian structure  $\langle \cdot, \cdot \rangle$  on  $H^{(k)}$  also induces the operator norm on  $\text{End}(H^{(k)})$ . By the finite dimensionality of  $H_\sigma^{(k)}$  in  $\mathcal{H}_\sigma^{(k)}$  we have the orthogonal projection  $\pi_\sigma^{(k)} : \mathcal{H}_\sigma^{(k)} \rightarrow H_\sigma^{(k)}$ . From these projections we can construct the Toeplitz operators associated to any smooth function  $f \in C^\infty(M)$ . It is the operator  $T_{f,\sigma}^{(k)} : \mathcal{H}_\sigma^{(k)} \rightarrow H_\sigma^{(k)}$  defined by

$$T_{f,\sigma}^{(k)}(s) = \pi_\sigma^{(k)}(fs)$$

for any element  $s \in \mathcal{H}_\sigma^{(k)}$  and any point  $\sigma \in \mathcal{T}$ . Since the projections form a smooth map  $\pi^{(k)}$  from  $\mathcal{T}$  to the space of bounded operators in the  $L_2$ -completion of  $C^\infty(M, \mathcal{L}^k)$  the Toeplitz operators are smooth sections  $T_f^{(k)}$  of the bundle of homomorphisms  $\text{Hom}(\mathcal{H}^{(k)}, H^{(k)})$  and restrict to smooth sections of  $\text{End}(H^{(k)})$ .

**Remark 4.** *It should be remarked that the above construction could be used for any Pseudo-differential operator  $A$  on  $M$  with coefficients in  $\mathcal{L}^k$  – it can even depend on  $\sigma$ , and we will then consider it as a section of  $\text{Hom}(\mathcal{H}^{(k)}, H^{(k)})$ . However when we consider their asymptotic expansions or operator norms, we implicitly restrict them to  $H^{(k)}$  and consider them as sections of  $\text{End}(H^{(k)})$  – or as  $\pi^{(k)} A \pi^{(k)}$ .*

We need the following two theorems on Toeplitz operators to proceed. The first is due to Bordemann, Meinrenken and Schlichenmaier (see [BMS]).

**Theorem 6** (Bordemann, Meinrenken and Schlichenmaier). *For any  $f \in C^\infty(M)$  we have that*

$$\lim_{k \rightarrow \infty} \|T_f^{(k)}\| = \sup_{x \in M} |f(x)|.$$

Since the association of the sequence of Toeplitz operators  $T_f^{(k)}$ ,  $k \in \mathbb{Z}_+$  is linear in  $f$ , we see from this Theorem, that this association is faithful.

The product of two Toeplitz operators associated to two smooth functions will in general not be a Toeplitz operator associated to a smooth function again. But by Schlichenmaier [Sch], there is an asymptotic expansion of the product in terms of Toeplitz operators associated to smooth functions on a compact Kähler manifold.

**Theorem 7** (Schlichenmaier). *For any pair of smooth functions  $f_1, f_2 \in C^\infty(M)$ , we have an asymptotic expansion*

$$T_{f_1}^{(k)} T_{f_2}^{(k)} \sim \sum_{l=0}^{\infty} T_{c_l(f_1, f_2)}^{(k)} k^{-l},$$

where  $c_l(f_1, f_2) \in C^\infty(M)$  are uniquely determined since  $\sim$  means the following: For all  $L \in \mathbb{Z}_+$  we have that

$$(7) \quad \|T_{f_1}^{(k)} T_{f_2}^{(k)} - \sum_{l=0}^L T_{c_l(f_1, f_2)}^{(k)} k^{-l}\| = O(k^{-(L+1)})$$

uniformly over compact subsets of  $\mathcal{T}$ . Moreover,  $c_0(f_1, f_2) = f_1 f_2$ .

**Remark 5.** In Section 5 it will be useful for us to define new coefficients  $\tilde{c}_\sigma^{(l)}(f, g) \in C^\infty(M)$  which correspond to the expansion of the product in  $1/(k + n/2)$  (where  $n$  is some fixed integer):

$$T_{f_1, \sigma}^{(k)} T_{f_2, \sigma}^{(k)} \sim \sum_{l=0}^{\infty} T_{\tilde{c}_\sigma^{(l)}(f_1, f_2), \sigma}^{(k)} (k + n/2)^{-l}.$$

Note that the first three coefficients are given by  $\tilde{c}_\sigma^{(0)}(f_1, f_2) = c_\sigma^{(0)}(f_1, f_2)$ ,  $\tilde{c}_\sigma^{(1)}(f_1, f_2) = c_\sigma^{(1)}(f_1, f_2)$  and  $\tilde{c}_\sigma^{(2)}(f_1, f_2) = c_\sigma^{(2)}(f_1, f_2) + \frac{n}{2} c_\sigma^{(1)}(f_1, f_2)$ .

This Theorem was proved in [Sch] where it is also proved that the formal generating series for the  $c_l(f_1, f_2)$ 's gives a formal deformation quantization of the Poisson structure on  $M$  induced by  $\omega$ . An English version is available in [Sch1] see [Sch2] for further developments. We return to this in Section 5 where we discuss formal Hitchin connections.

#### 4. ASYMPTOTIC FAITHFULNESS

In this section we will concentrate on the case where  $M$  is the moduli space of flat  $\mathrm{SU}(n)$ -connections on  $\Sigma - p$  with holonomy  $d$  around  $p$ . As in the introduction  $\Sigma$  is a closed oriented surface of genus  $g \geq 2$ ,  $p$  a point in  $\Sigma$  and  $d \in \mathbb{Z}/n\mathbb{Z} \simeq Z_{\mathrm{SU}(n)}$  in the center of  $\mathrm{SU}(n)$  is fixed.

As mentioned in the introduction the main result about the Verlinde bundle  $\mathcal{V}^{(k)}$  from geometrically quantizing the moduli space  $M$  is that its projectivization  $\mathbb{P}(\mathcal{V}^{(k)})$  carries a flat connection  $\hat{\nabla}$ . This flat connection induces a flat connection in  $\hat{\nabla}^e$  in the endomorphism bundle  $\mathrm{End}(\mathcal{V}^{(k)})$  as described in Section 2.

An important ingredient in proving asymptotic faithfulness is the corollary to Theorem 4 saying that Toeplitz operators viewed as a section of  $\text{End}(\mathcal{V}^{(k)})$  is in some sense asymptotically flat,

$$\|\hat{\nabla}_V^e T_f^{(k)}\| = O(k^{-1}).$$

This can be reformulated in terms of the induced parallel transport between the fibers of  $\text{End}(\mathcal{V}^{(k)})$ . Let  $\sigma_0, \sigma_1$  be two points in Teichmüller space  $\mathcal{T}$ , and  $P_{\sigma_0, \sigma_1}$  the parallel transport from  $\sigma_0$  to  $\sigma_1$ . Then

$$(8) \quad \|P_{\sigma_0, \sigma_1} T_{f, \sigma_0}^{(k)} - T_{f, \sigma_1}^{(k)}\| = O(k^{-1}),$$

where  $\|\cdot\|$  is the operator norm on  $H^0(M_{\sigma_1}, \mathcal{L}_{\sigma_1}^k)$ .

Equation (8) and Theorem 6 together prove asymptotic faithfulness. Below we explain how.

Recall that the flat connection in the bundle  $\mathbb{P}(\mathcal{V}^{(k)})$  gives the projective representation of the mapping class group

$$Z_k^{(n, d)} : \Gamma \rightarrow \text{Aut}(\mathbb{P}(V^k))$$

where  $\mathbb{P}(V^{(k)})$  are the covariant constant sections of  $\mathbb{P}(\mathcal{V}^{(k)})$  over Teichmüller space with respect to the Hitchin connection  $\hat{\nabla}$ .

*Proof of Theorem 2.* Suppose we have a  $\phi \in \Gamma$ . Then  $\phi$  induces a symplectomorphism of  $M$  which we also just denote  $\phi$  and we get the following commutative diagram for any  $f \in \mathbb{C}^\infty(M)$

$$\begin{array}{ccccc} H^0(M_\sigma, \mathcal{L}_\sigma^k) & \xrightarrow{\phi^*} & H^0(M_{\phi(\sigma)}, \mathcal{L}_{\phi(\sigma)}^k) & \xrightarrow{P_{\phi(\sigma), \sigma}} & H^0(M_\sigma, \mathcal{L}_\sigma^k) \\ T_{f, \sigma}^{(k)} \downarrow & & T_{f \circ \phi, \phi(\sigma)}^{(k)} \downarrow & & \downarrow P_{\phi(\sigma), \sigma} T_{f \circ \phi, \phi(\sigma)}^{(k)} \\ H^0(M_\sigma, \mathcal{L}_\sigma^k) & \xrightarrow{\phi^*} & H^0(M_{\phi(\sigma)}, \mathcal{L}_{\phi(\sigma)}^k) & \xrightarrow{P_{\phi(\sigma), \sigma}} & H^0(M_\sigma, \mathcal{L}_\sigma^k), \end{array}$$

where  $P_{\phi(\sigma), \sigma} : H^0(M_{\phi(\sigma)}, \mathcal{L}_{\phi(\sigma)}^k) \rightarrow H^0(M_\sigma, \mathcal{L}_\sigma^k)$  on the horizontal arrows refer to parallel transport in the Verlinde bundle itself, whereas  $P_{\phi(\sigma), \sigma}$  refers to the parallel transport in the endomorphism bundle  $\text{End}(\mathcal{V}_k)$  in the last vertical arrow. Suppose now  $\phi \in \bigcap_{k=1}^\infty \ker Z_k^{(n, d)}$ , then  $P_{\phi(\sigma), \sigma} \circ \phi^* = Z_k^{(n, d)}(\phi) \in \mathbb{C}\text{Id}$  and we get that  $T_{f, \sigma}^{(k)} = P_{\phi(\sigma), \sigma} T_{f \circ \phi, \phi(\sigma)}^{(k)}$ . By Theorem 8 we get that

$$\begin{aligned} \lim_{k \rightarrow \infty} \|T_{f - f \circ \phi, \sigma}^{(k)}\| &= \lim_{k \rightarrow \infty} \|T_{f, \sigma}^{(k)} - T_{f \circ \phi, \sigma}^{(k)}\| \\ &= \lim_{k \rightarrow \infty} \|P_{\phi(\sigma), \sigma} T_{f \circ \phi, \phi(\sigma)}^{(k)} - T_{f \circ \phi, \sigma}^{(k)}\| = 0. \end{aligned}$$

By Bordemann, Meinrenken and Schlichenmaier's Theorem 6, we must have that  $f = f \circ \phi$ . Since this holds for any  $f \in \mathbb{C}^\infty(M)$ , we must have that  $\phi$  acts by the identity on  $M$ .  $\square$

## 5. THE FORMAL HITCHIN CONNECTION AND BEREZIN–TOEPLITZ DEFORMATION QUANTIZATION

In this section we discuss the formal Hitchin connection. We return to the general setup of compact Kähler manifolds, where we impose conditions on  $(M, \omega, I)$  as in Theorem 5, thus providing us with a Hitchin connection  $\hat{\nabla}$  in  $H^{(k)}$  over  $\mathcal{T}$  and the associated connection  $\hat{\nabla}^e$  in  $\text{End}(H^{(k)})$ . Firstly we recall the definition of a

formal deformation quantization and the results about star products from [Sch] and [KS]. We introduce the space of formal functions  $C_h^\infty(M) = C^\infty(M)[[h]]$  as the space for formal power series in the variable  $h$  with coefficients in  $C^\infty(M)$ , and let  $\mathbb{C}_h = \mathbb{C}[[h]]$  denote the formal constants.

**Definition 8.** *A deformation quantization of  $(M, \omega)$  is an associative product  $\star$  on  $C_h^\infty(M)$  which respects the  $\mathbb{C}_h$ -module structure. For  $f, g \in C^\infty(M)$ , it is defined as*

$$f \star g = \sum_{l=0}^{\infty} c^{(l)}(f, g) h^l,$$

through a sequence of bilinear operators

$$c^{(l)} : C^\infty(M) \otimes C^\infty(M) \rightarrow C^\infty(M),$$

which must satisfy

$$c^{(0)}(f, g) = fg \quad \text{and} \quad c^{(1)}(f, g) - c^{(1)}(g, f) = -i\{f, g\}.$$

The deformation quantization is said to be differential if the operators  $c^{(l)}$  are bidifferential operators. Considering the symplectic action of  $\Gamma$  on  $(M, \omega)$ , we say that a star product is  $\Gamma$ -invariant if

$$\gamma^*(f \star g) = \gamma^*(f) \star \gamma^*(g)$$

for all  $f, g \in C^\infty(M)$  and all  $\gamma \in \Gamma$ .

Recall Theorem 7 where the asymptotic expansion of the product of two Toeplitz operators associated to smooth functions  $f_1, f_2$  on  $M$  create maps  $c_i(f_1, f_2) \in C^\infty(M)$ . In [Sch] Schlichenmaier also showed that these maps generate a star product. It was first in [KS] Karabegov and Schlichenmaier showed that it was a differentiable star product.

**Theorem 8** (Karabegov & Schlichenmaier). *The product  $\star_\sigma^{BT}$  given by*

$$f \star_\sigma^{BT} g = \sum_{l=0}^{\infty} c_\sigma^{(l)}(f, g) h^l,$$

where  $f, g \in C^\infty(M)$  and  $c_\sigma^{(l)}(f, g)$  are determined by Theorem 7, is a differentiable deformation quantization of  $(M, \omega)$ .

**Definition 9.** *The Berezin-Toeplitz deformation quantization of the compact Kähler manifold  $(M_\sigma, \omega)$  is the product  $\star_\sigma^{BT}$ .*

For the remaining part of this paper we let  $\Gamma$  be a symmetry group as in Section 2, that is a group which acts by bundle automorphisms on  $\mathcal{L}$  over  $M$  preserving both the Hermitian structure and the connection in  $\mathcal{L}$ . Such a group has an induced action on  $(M, \omega)$ . Note that  $\Gamma$  in the case of moduli spaces is the mapping class group of the surface.

**Remark 6.** *Let  $\Gamma_\sigma$  be the  $\sigma$ -stabilizer subgroup of  $\Gamma$ . For any element  $\gamma \in \Gamma_\sigma$ , we have that*

$$\gamma^*(T_{f, \sigma}^{(k)}) = T_{\gamma^* f, \sigma}^{(k)}.$$

This implies the invariance of  $\star_\sigma^{BT}$  under the  $\sigma$ -stabilizer  $\Gamma_\sigma$ .

**Remark 7.** Using the coefficients from Remark 5, we define a new star product by

$$f \tilde{\star}_\sigma^{BT} g = \sum_{l=0}^{\infty} \tilde{c}_\sigma^{(l)}(f, g) h^l.$$

Then

$$f \tilde{\star}_\sigma^{BT} g = ((f \circ \phi^{-1}) \star_\sigma^{BT} (g \circ \phi^{-1})) \circ \phi$$

for all  $f, g \in C_h^\infty(M)$ , where  $\phi(h) = \frac{2h}{2+n}$ .

Recall from the introduction the definition of a formal connection in the trivial bundle of formal functions. Theorem 4, establishes the existence of a unique formal Hitchin connection, expressing asymptotically the interplay between the Hitchin connection and the Toeplitz operators.

We want to give an explicit formula for the formal Hitchin connection in terms of the star product  $\tilde{\star}^{BT}$ . We recall that in the proof of Theorem 4, given in [A6], it is shown that the formal Hitchin connection is given by

$$(9) \quad \tilde{D}(V)(f) = -V[F]f + V[F]\tilde{\star}^{BT}f + h(E(V)(f) - H(V)\tilde{\star}^{BT}f),$$

where  $E$  is the one-form on  $\mathcal{T}$  with values in  $\mathcal{D}(M)$  such that

$$(10) \quad T_{E(V)f}^{(k)} = \pi^{(k)} o(V)^* f \pi^{(k)} + \pi^{(k)} f o(V) \pi^{(k)},$$

and  $H$  is the one form on  $\mathcal{T}$  with values in  $C^\infty(M)$  such that  $H(V) = E(V)(1)$ . In [AG] an explicit expression for the operator  $E(V)$  is found by calculating the adjoint of

$$o(V) = -\frac{1}{4}(\Delta_{G(V)} + 2\hat{\nabla}_{G(V) \cdot dF} - 2nV'[F]).$$

This operator is essential in the proof of Theorem 5 since by comparing the above equation with Equation 5 we see that  $u(V) = \frac{1}{k+n/2}o(V) - V'[F]$ .

**Theorem 9.** The formal Hitchin connection is given by

$$\begin{aligned} D_V f &= V[f] - \frac{1}{4}h\Delta_{\tilde{G}(V)}(f) + \frac{1}{2}h\nabla_{\tilde{G}(V)dF}(f) + V[F]\tilde{\star}^{BT}f - V[F]f \\ &\quad - \frac{1}{2}h(\Delta_{\tilde{G}(V)}(F)\tilde{\star}^{BT}f + nV[F]\tilde{\star}^{BT}f - \Delta_{\tilde{G}(V)}(F)f - nV[F]f) \end{aligned}$$

for any vector field  $V$  and any section  $f$  of  $C_h$ .

When we geometrically quantize a symplectic manifold, we have to choose a polarization of the complexified tangent bundle, to reduce the space upon the quantum operators act. This is equivalent to choosing a compatible complex structure on the symplectic manifold, hence making it Kähler. It is however quite unfortunate that the quantum space then depend on the choice of Kähler structure. The solution to this is the projectively flat Hitchin Connection, which by parallel transport between the fibers of  $H^{(k)}$  give us a space of quantum states as the covariant constant sections of  $\mathbb{P}H^{(k)}$ , which does not depend on the chosen complex structure. Instead of doing geometric quantization we could do Berezin–Toeplitz deformation quantization. The created star product  $\star_\sigma^{BT}$  depend on the complex structure, and in the same spirit as above we want to make all these star products equivalent to a star product which does not depend on  $\sigma$ . This is the purpose of the formal Hitchin connection.

If the Hitchin connection is projectively flat, then the induced connection in the endomorphism bundle is flat and hence so is the formal Hitchin connection by Proposition 3 of [A6].

Recall from Definition 3 in the introduction the definition of a formal trivialization. As mentioned there, such a formal trivialization will not exist even locally on  $\mathcal{T}$ , if  $D$  is not flat. However, if  $D$  is flat, then we have the following result from [A6].

**Proposition 1.** *Assume that  $D$  is flat and that  $\tilde{D} = 0 \bmod h$ . Then locally around any point in  $\mathcal{T}$ , there exists a formal trivialization. If  $H^1(\mathcal{T}, \mathbb{R}) = 0$ , then there exists a formal trivialization defined globally on  $\mathcal{T}$ . If further  $H_\Gamma^1(\mathcal{T}, D(M)) = 0$ , then we can construct  $P$  such that it is  $\Gamma$ -equivariant.*

An immediate corollary of Proposition 1 is

**Corollary 1.** *If  $\mathcal{T}$  is contractible, then any flat formal connection admits a global formal trivialization that is  $\Gamma$ -equivariant.*

In the proposition,  $H_\Gamma^1(\mathcal{T}, D(M))$  refers to the  $\Gamma$ -equivariant first de Rham cohomology of  $\mathcal{T}$  with coefficients in the real vector space  $D(M)$  of differential operators on  $M$ . The first steps towards proving that this cohomology group vanishes in the case where  $M$  is the moduli space have been taken in [AV1, AV2, AV3, Vi].

In [AG] an explicit formula for  $P$  up to first order is found.

**Theorem 10.** *The  $\Gamma$ -equivariant formal trivialization of the formal Hitchin connection exists to first order, and we have the following explicit formula for the first order term of  $P$*

$$P_\sigma^{(1)}(f) = \frac{1}{4}\Delta_\sigma(f) + i\nabla_{X_F''}(f),$$

where  $X_F''$  denotes the  $(0,1)$ -part of the Hamiltonian vector field for the Ricci potential,  $F$ .

Now suppose we have a formal trivialization  $P$  of the formal Hitchin connection  $D$ . We can then define a new smooth family of star products, parametrized by  $\mathcal{T}$ , by

$$f \star_\sigma g = P_\sigma^{-1}(P_\sigma(f) \tilde{\star}_\sigma^{BT} P_\sigma(g))$$

for all  $f, g \in C^\infty(M)$  and all  $\sigma \in \mathcal{T}$ . Using the fact that  $P$  is a trivialization, it is not hard to prove

**Proposition 2.** *The star products  $\star_\sigma$  are independent of  $\sigma \in \mathcal{T}$ .*

This is done by simply differentiating  $\star_\sigma$  along a vector field on  $\mathcal{T}$ , see [A6].

Then, we have the following which is proved in [A6].

**Theorem 11** (Andersen). *Assume that the formal Hitchin connection  $D$  is flat and*

$$H_\Gamma^1(\mathcal{T}, D(M)) = 0,$$

*then there is a  $\Gamma$ -invariant trivialization  $P$  of  $D$  and the star product*

$$f \star g = P_\sigma^{-1}(P_\sigma(f) \tilde{\star}_\sigma^{BT} P_\sigma(g))$$

*is independent of  $\sigma \in \mathcal{T}$  and  $\Gamma$ -invariant. If  $H_\Gamma^1(\mathcal{T}, C^\infty(M)) = 0$  and the commutant of  $\Gamma$  in  $D(M)$  is trivial, then a  $\Gamma$ -invariant differential star product on  $M$  is unique.*



In [AG] the star product of Theorem 11 is identified up to second order in  $h$ .

**Theorem 12.** *The star product  $\star$  has the form*

$$f \star g = fg - \frac{i}{2}\{f, g\}h + O(h^2).$$

We observe that this formula for the first-order term of  $\star$  agrees with the first-order term of the star product constructed by Andersen, Mattes and Reshetikhin in [AMR2], when we apply the formula in Theorem 12 to two holonomy functions  $h_{\gamma_1, \lambda_1}$  and  $h_{\gamma_2, \lambda_2}$ :

$$h_{\gamma_1, \lambda_1} \star h_{\gamma_2, \lambda_2} = h_{\gamma_1 \gamma_2, \lambda_1 \cup \lambda_2} - \frac{i}{2}h_{\{\gamma_1, \gamma_2\}, \lambda_1 \cup \lambda_2} + O(h^2).$$

We recall that  $\{\gamma_1, \gamma_2\}$  is the Goldman bracket (see [Go2]) of the two simple closed curves  $\gamma_1$  and  $\gamma_2$ .

A similar result was obtained for the abelian case, i.e. in the case where  $M$  is the moduli space of flat  $U(1)$ -connections, by the first author in [A2], where the agreement between the star product defined in differential geometric terms and the star product of Andersen, Mattes and Reshetikhin was proved to all orders.

## 6. ABELIAN VARIETIES AND $U(1)$ -MODULI SPACE

In this chapter we will investigate all the previous mentioned objects in the setting of principally polarized abelian varieties  $M = V/\Lambda$ , where  $V$  is a real vector space with a symplectic form  $\omega$ ,  $\Lambda$  a discrete lattice of maximal rank such that  $\omega$  is integral and unimodular when restricted to  $\Lambda$ . Let now  $\mathcal{T}$  be the space of complex structures on  $V$ , which are compatible with  $\omega$ . Then for any  $I \in \mathcal{T}$ ,  $M_I = (M, \omega, I)$  is an abelian variety. A prime example of an abelian variety is the abelian moduli space. Here we let  $\Sigma$  be a closed surface of genus  $g$ , and  $M$  be the moduli space of flat  $U(1)$ -connections on  $\Sigma$ . Then

$$M = \text{Hom}(\pi_1(\Sigma), U(1)) = H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z}).$$

There is the usual symplectic structure  $\omega$  on  $H^1(M, \mathbb{R})$  which is of course integral and unimodular over the lattice  $H^1(M, \mathbb{Z})$ . We will return to this example below when we consider abelian Chern–Simons theory.

In the following we will focus on  $M_I$  being a principal polarized abelian variety, where the compatible complex structures are parametrized by  $\mathcal{T}$ . There exists a symplectic basis  $(\lambda_1, \dots, \lambda_{2n})$  over the integers for  $\Lambda$  (e.g. [GH, p. 304]). Let  $(x_1, \dots, x_n, y_1, \dots, y_{2n})$  be the dual coordinates on  $V$ . Then

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i.$$

Let  $A$  be the automorphism group of  $(\Lambda, \omega)$ . Then  $A$  injects into the symplectomorphism group of  $(M, \omega)$ , and by using the symplectic basis  $(\lambda_1, \dots, \lambda_{2n})$  we get an identification  $A \simeq \text{Sp}(2n, \mathbb{Z})$ . Notice that  $A$  acts on the principal polarized variety  $M_I$ .

Using the symplectic basis we can identify  $\mathcal{T}$  with the Siegel Upper Half Space

$$\mathbb{H} = \{Z \in M_{n,n}(\mathbb{C}) \mid Z = Z^T, \text{Im}(Z) > 0\}.$$

For any  $I \in \mathcal{T}$  we have that  $(\lambda_1, \dots, \lambda_n)$  is a basis over  $\mathbb{C}$  for  $V$  with respect to  $I$ . Let  $(z_1, \dots, z_n)$  be the dual complex coordinates on  $V$  relative to the basis

$(\lambda_1, \dots, \lambda_n)$ . The complex structure  $I$  determines and is determined by a unique  $Z \in \mathbb{H}$  such that

$$z = x + Zy.$$

Since any  $Z \in \mathbb{H}$  gives a complex structure, say  $I(Z)$ , compatible with the symplectic form, we have a bijective map  $I : \mathbb{H} \rightarrow \mathcal{T}$  given by sending  $Z \in \mathbb{H}$  to  $I(Z)$ . For  $Z \in \mathbb{H}$  we use the notation  $X = \operatorname{Re}(Z)$  and  $Y = \operatorname{Im}(Z)$ .

For each  $Z \in \mathbb{H}$  we explicitly construct a prequantum line bundle on  $M_{I(Z)}$ . We do that by providing a lift of the action  $\Lambda$  action on  $V$  to the trivial bundle  $\tilde{\mathcal{L}} = V \times \mathbb{C}$ , such that the quotient is the prequantum line bundle  $\mathcal{L}_Z$ . We only need to specify a set of multipliers  $\{e_\lambda\}_{\lambda \in \Lambda}$  and a Hermitian structure  $h$ . The multipliers are non-vanishing functions on  $V$  that are holomorphic with respect to  $I(Z)$  and depend on  $Z$ . They should furthermore satisfy the following functional equation

$$e_{\lambda'}(v + \lambda)e_\lambda(v) = e_{\lambda'}(v)e_\lambda(v + \lambda') = e_{\lambda + \lambda'}(v),$$

for all  $\lambda, \lambda' \in \Lambda$ . The action of  $\Lambda$  on  $\tilde{\mathcal{L}}$  is given by

$$\lambda \cdot (v, z) = (v + \lambda, e_\lambda(z)),$$

for all  $\lambda \in \Lambda$  and  $(v, z) \in \tilde{\mathcal{L}}$ . For a fixed basis of  $\Lambda$  the functional equations determine the multipliers for all  $\lambda \in \Lambda$ . For  $I(Z)$  we choose the multipliers

$$\begin{aligned} e_{\lambda_i}(z) &= 1, & i &= 1, \dots, n, \\ e_{\lambda_{n+i}}(z) &= e^{-2\pi i z_i - \pi i Z_{ii}}, & i &= 1, \dots, n. \end{aligned}$$

The constructed line bundle is denoted  $\mathcal{L}_Z$ . If we define  $h(z) = e^{-2\pi y \cdot Y y}$ , where  $Z = X + iY$ , it will define a Hermitian structure on  $V \times \mathbb{C}$  by  $h(z) \langle \cdot, \cdot \rangle_{\mathbb{C}}$  where  $\langle \cdot, \cdot \rangle_{\mathbb{C}}$  is the standard inner product on  $\mathbb{C}^n$ . This function satisfies the functional equation

$$h(z + \lambda) = \frac{1}{|e_\lambda(z)|^2} h(z),$$

and the inner product on  $V \times \mathbb{C}$  is invariant under the action of  $\Lambda$  and hence induces a Hermitian structure  $\langle \cdot, \cdot \rangle$  on  $\mathcal{L}_Z$ . By general theory of abelian varieties, e.g. [GH, Sect. 2.6], a line bundle with the above multipliers and Hermitian metric  $(\mathcal{L}_Z, \langle \cdot, \cdot \rangle)$  has curvature  $-2\pi i \omega$ , and hence is a prequantum line bundle. Note that the prequantum condition in Definition 4 is scaled with  $2\pi$ . We could just have used  $2\pi\omega$  as the symplectic structure. We choose the normalization at hand to make later equations nicer.

The space of holomorphic sections of  $\mathcal{L}_Z^k$ ,  $H^0(M_Z, \mathcal{L}_Z^k)$  has dimension  $k^n$ , and as in the general theory they give a vector bundle  $H^{(k)}$  over  $\mathbb{H}$  by letting  $H_Z^{(k)} = H^0(M_Z, \mathcal{L}_Z^k)$ .

The  $L^2$ -inner product on  $H^0(M_Z, \mathcal{L}_Z^k)$  is given by

$$(s_1, s_2) = \int_{M_Z} s_1(z) \overline{s_2(z)} h(z) dx dy,$$

for  $s_1, s_2 \in H^0(M_Z, \mathcal{L}_Z^k)$ .

A basis for the space of sections are the *Theta functions*,

$$\theta_\alpha^{(k)}(z, Z) = \sum_{l \in \mathbb{Z}^n} e^{\pi i k(l + \alpha) \cdot Z(l + \alpha)} e^{2\pi i k(l + \alpha) \cdot z},$$

where  $\alpha \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n$ . The Theta functions satisfies the following heat equation,

$$\frac{\partial \theta_\alpha^{(k)}}{\partial Z_{ij}} = \frac{1}{4\pi i k} \frac{\partial^2 \theta_\alpha^{(k)}}{\partial z_i \partial z_j}.$$

The geometric interpretation of this differential equation is a definition of a connection  $\tilde{\nabla}$  in the trivial  $C^\infty(\mathbb{C}^n)$ -bundle over  $\mathbb{H}$ , by

$$\tilde{\nabla}_{\frac{\partial}{\partial Z_{ij}}} = \frac{\partial}{\partial Z_{ij}} - \frac{1}{4\pi i k} \frac{\partial^2}{\partial z_i \partial z_j}.$$

The coordinates  $z = x + Zy$  identify  $H^0(M_Z, \mathcal{L}_Z^k)$  as a subspace of  $C^\infty(\mathbb{C}^n)$  and  $H^{(k)}$  as a subbundle of the trivial  $C^\infty(\mathbb{C}^n)$ -bundle on  $\mathbb{H}$ . This bundle is preserved by  $\tilde{\nabla}$  and hence induces a connection  $\nabla$  in  $H^{(k)}$ . The covariant constant sections of  $H^{(k)}$  with respect to  $\nabla$  will, under the embedding induced by the coordinates, be identified with the Theta functions. Since now  $\nabla$  has a global frame of covariant constant sections it is flat. Remember that  $\mathbb{H}$  is contractible, so since parallel transport with a flat connection only depend on the homotopy class of the curve transported along, we get a canonical way to identify all  $H^0(M_Z, \mathcal{L}_Z^k)$ , and hence there is no ambiguity in defining the quantum space of geometric quantization to be  $H^0(M_Z, \mathcal{L}_Z^k)$ . Since the Theta functions are covariant constant, they explicitly realize this identification. The usual action of  $\mathrm{Sp}(2n, \mathbb{Z})$  on Theta functions induce an action of  $A' = \ker(\mathrm{Sp}(2n, \mathbb{Z}) \rightarrow \mathrm{Sp}(2n, \mathbb{Z}/2\mathbb{Z}))$  on the bundle  $H^{(k)}$  which covers the  $A'$ -action on  $\mathbb{H} \simeq \mathcal{T}$ . This is the subgroup of  $A$  acting trivially on  $\Lambda/2\Lambda$ .

**Remark 8.** *Instead of the above connection  $\nabla$  in  $H^{(k)}$  over  $\mathbb{H}$ , we could have rolled out the machinery of Theorem 5 to get another connection in the same bundle. This can be done even though  $H^1(M, \mathbb{R}) \neq 0$ . Since the torus is flat the Ricci potential  $F$  is 0 as is the Chern class of  $M$ . Lemma 5 in the appendix shows that  $I(Z)$  is constant on  $M$  and thus is a rigid family of Kähler structures. Thus we have a rather nice formula for the Hitchin connection*

$$\hat{\nabla}_V = \nabla_V^t + \frac{1}{8\pi k} \Delta_{G(V)}.$$

*The extra factor of  $2\pi$  is from the different prequantum condition. It should be noted that explicit computations show that  $\hat{\nabla}$  is not flat like  $\nabla$  induced by the heat equation, but rather projectively flat.*

In [A2] the inner product of two Theta functions are explicitly calculated.

**Lemma 3.** *The theta functions  $\theta_\alpha^{(k)}(z, Z)$ ,  $\alpha \in \frac{1}{k}\mathbb{Z}^n/\mathbb{Z}^n$ , define an orthonormal basis with respect to the inner product on  $H^0(M_Z, \mathcal{L}_Z^k)$  defined by*

$$(s_1, s_2)_Y = (s_1, s_2) \sqrt{2^n k^n \det Y},$$

*where  $Y = \mathrm{Im} Z$ . This is a Hermitian structure on  $H^{(k)}$  compatible with  $\nabla$ .*

Let  $(r, s) \in \mathbb{Z}^n \times \mathbb{Z}^n$  and consider the function  $F_{r,s} \in C^\infty(M)$  given in  $(x, y)$ -coordinates by

$$F_{r,s}(x, y) = e^{2\pi i(x \cdot r + s \cdot y)}.$$

We have previously defined Toeplitz operators associated to a function  $f \in C^\infty(M)$ ,  $T_f^{(k)} : H^0(M_Z, \mathcal{L}_Z^k) \rightarrow H^0(M_Z, \mathcal{L}_Z^k)$ . We shall now explicitly compute the matrix coefficients of these operators in terms of the basis consisting of Theta functions.

To get our hands on the matrix coefficients  $(T_{F_{r,s}}^{(k)})_{\beta\alpha}$  we only need to calculate  $(F_{r,s}\theta_\alpha^{(k)}, \theta_\beta^{(k)})$ , since this indeed is the coefficient. This is also calculated in [A2] and is done in the exact same way as in Lemma 3,

$$(11) \quad (F_{r,s}\theta_\alpha^{(k)}, \theta_\beta^{(k)})_Y = \delta_{\alpha-\beta, -[\frac{r}{k}]} e^{-\frac{\pi i}{k} r \cdot \bar{Z}r} e^{-2\pi i s \cdot \alpha} e^{-\pi^2 (s - \bar{Z}r) \cdot (2\pi k Y)^{-1} (s - \bar{Z}r)},$$

where  $[\frac{r}{k}]$  is the residue class of  $\frac{r}{k}$  mod  $\mathbb{Z}^n$ . A simple rewriting gives

$$(T_{f(r,s,Z)(k)F_{r,s}}^{(k)})_{\beta\alpha} = \delta_{\alpha-\beta, -[\frac{r}{k}]} e^{-\frac{\pi i}{k} r \cdot s} e^{-2\pi i s \cdot \alpha},$$

where

$$f(r, s, Z)(k) = e^{\frac{\pi}{2k} (s - Xr) \cdot Y^{-1} (s - Xr)} e^{\frac{\pi}{2k} r \cdot Yr}.$$

**Remark 9.** The Toeplitz operators  $T_{F_{r,s}}^{(k)}$  are sections of  $\text{End}(H^{(k)})$  over  $\mathbb{H}$ . The flat connection  $\nabla$  induces a flat connection  $\nabla^e$  in the bundle  $\text{End}(H^{(k)})$ , with respect to which we see that  $T_{F_{r,s}}^{(k)}$  is not covariant constant. However the operators  $T_{f(r,s,Z)(k)F_{r,s}}^{(k)}$  are covariant constant. Since the pure phases  $F_{r,s}$ ,  $r, s \in \mathbb{Z}^n$  is a Fourier basis for  $C^\infty(M)$ , we have that  $T_{f(r,s,Z)(k)F_{r,s}}^{(k)}$  is covariant constant with respect to  $\hat{\nabla}^e$  for all  $f \in C^\infty(M)$ .

It should also be noted that the coefficient  $f(r, s, Z)(k)$  is not so arbitrary as it looks. This is the content of the following

**Proposition 3.** Let  $\Delta_{I(Z)}$  be the Laplace operator with respect to the metric

$$g_{I(Z)}(\cdot, \cdot) = 2\pi\omega(\cdot, I(Z)\cdot)$$

on  $M$ . Then

$$e^{-\frac{1}{4k}\Delta_{I(Z)}} F_{r,s} = f(r, s, Z)(k) F_{r,s}.$$

*Proof.* Recall that

$$\Delta_{I(Z)} = \frac{1}{2\pi} \left( \left( \frac{\partial}{\partial y} - X \frac{\partial}{\partial x} \right) \cdot Y^{-1} \left( \frac{\partial}{\partial y} - X \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial x} \cdot Y \frac{\partial}{\partial x} \right).$$

Now it is a simple calculation, which we will omit, to show the equality.  $\square$

As remarked in Remark 9,  $T_{f(r,s,Z)(k)F_{r,s}}^{(k)}$  is covariant constant with respect to  $\hat{\nabla}$ . If we define

$$E_{I(Z)} = e^{-\frac{h}{4}\Delta_{I(Z)}} : C_h^\infty(M) \rightarrow C_h^\infty(M)$$

we see that

$$\hat{\nabla}_V^e T_{E_I(f)(1/k)}^{(k)} = 0,$$

for all vector fields  $V$  on  $\mathbb{H}$  and all functions  $f \in C^\infty(M)$  since the pure phase functions constitute a Fourier basis. We furthermore see that  $E_I$  is  $\text{Sp}(2n, \mathbb{Z})$ -equivariant, since for all  $\Psi \in \text{Sp}(2n, \mathbb{Z})$  we have that

$$\Psi^* \circ E_I = E_{\Psi(I)} \Psi^*.$$

That  $T_{E_I(f)}^{(k)}$  is covariant constant with respect to  $\hat{\nabla}^e$  can be interpreted as  $E_I$  is a formal parametrization for the formal Hitchin Connection which we know exists by Theorem 4. If  $\hat{\nabla}_V^e T_{E_I(f)}^{(k)} = 0$  Equation (2) and Theorem 6 imply that

$$D_V(E_I(f)) = 0$$

for all vector fields on  $\mathbb{H}$  and all  $f \in C_h^\infty(M)$ , so by Definition 3,  $E_I$  is a formal trivialization of the formal connection  $D$ . We compare this with the explicit formula for the first order term of  $P$  in Theorem 10, and see that they agree since the Ricci potential  $F$  is 0.

Now since the Ricci potential is 0 we reduce the formula in Theorem 9 for the formal Hitchin connection.

**Theorem 13.** *Let  $(M, \omega, I(Z))$  be a principal polarized variety, then the formal Hitchin connection is given by*

$$D_V f = V[f] - \frac{1}{8\pi} h \Delta_{\tilde{G}(V)}(f),$$

and if  $Z$  is normal, we get explicit formulas for  $\Delta_{\tilde{G}(V)}$ . If  $i \neq j$

$$\Delta_{\tilde{G}(\frac{\partial}{\partial z_{ij}})} = 2i \nabla_{\frac{\partial}{\partial z_i}} \nabla_{\frac{\partial}{\partial z_j}} + 2i \nabla_{\frac{\partial}{\partial z_j}} \nabla_{\frac{\partial}{\partial z_i}} \quad \text{and} \quad \Delta_{\tilde{G}(\frac{\partial}{\partial \bar{z}_{ij}})} = 2i \nabla_{\frac{\partial}{\partial \bar{z}_i}} \nabla_{\frac{\partial}{\partial \bar{z}_j}} + 2i \nabla_{\frac{\partial}{\partial \bar{z}_j}} \nabla_{\frac{\partial}{\partial \bar{z}_i}},$$

and if  $i = j$

$$\Delta_{\tilde{G}(\frac{\partial}{\partial z_{ii}})} = 2i \nabla_{\frac{\partial}{\partial z_i}} \nabla_{\frac{\partial}{\partial z_i}} \quad \text{and} \quad \Delta_{\tilde{G}(\frac{\partial}{\partial \bar{z}_{ii}})} = 2i \nabla_{\frac{\partial}{\partial \bar{z}_i}} \nabla_{\frac{\partial}{\partial \bar{z}_i}}.$$

This theorem is proved in the appendix. It should be noted that the requirement on  $Z$  to be normal, only is to ease the calculations, and it will not be used anywhere else in the rest of this paper.

With this formal trivialization we use Theorem 11 and create an  $I$  independent star product on  $C^\infty(M)$  which all Berezin–Toeplitz star products are equivalent to. This is done in [A2] Theorem 5 where it is shown that the  $I$  independent star product actually is the Moyal–Weyl product

$$f \star g = \mu \circ \exp(-\frac{i}{2} h Q)(f \otimes g),$$

where  $\mu : C^\infty(\mathbb{R}^{2n}) \otimes C^\infty(\mathbb{R}^{2n}) \rightarrow C^\infty(\mathbb{R}^{2n})$  given by multiplication  $f \otimes g \mapsto fg$  and

$$Q = \sum_i \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_i} \otimes \frac{\partial}{\partial x_i}.$$

Again we see that this is exactly as in Theorem 12.

**Abelian Chern–Simons Theory.** In 2+1 dimensional Chern–Simons theory, the 2-dimensional part of the theory is a modular functor, which is a functor from the category of compact smooth oriented surfaces to the category of finite dimensional complex vector spaces, which satisfy certain properties. In the gauge-theoretic construction of this functor one first fixes a compact Lie group  $G$  and an invariant non-degenerate inner product on its Lie algebra. The functor then associates to a closed oriented surface the finite dimensional vector space one obtains by applying geometric quantization to the moduli space of flat  $G$ -connections on the surface (see e.g. [Wi] and [At]). In the abelian case  $G = U(1)$  at hand this concretely means the following. For a closed oriented surface  $\Sigma$  the moduli space of flat  $U(1)$ -connections

$$M = \text{Hom}(\pi_1(\Sigma), U(1)) = H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$$

has a symplectic structure given by the cup product followed by evaluation on the fundamental class of  $\Sigma$ . This symplectic structure is by Poincaré duality integral

and is unimodular over the lattice  $H^1(\Sigma, \mathbb{Z})$ . A subgroup of the mapping class group  $\Gamma$  of  $\Sigma$  acts on  $M$  via the induced homomorphism

$$\rho : \Gamma \rightarrow \text{Aut}(H^1(\Sigma, \mathbb{Z}), \omega) = \text{Sp}(2n, \mathbb{Z}).$$

Define  $\Gamma' = \rho^{-1}(A')$  and

$$\rho' = \rho|_{\Gamma'} : \Gamma' \rightarrow A'.$$

The homomorphism  $\rho'$  is surjective and has the Torelli subgroup of  $\Gamma$  as its kernel.

If we use the above theory we construct a Hermitian vector bundle  $H^{(k)}$  over the space of complex structures  $\mathcal{T}$  on  $H^1(\Sigma, \mathbb{R})$ . As discussed this bundle has a flat connection, and an action of  $\text{Aut}(H^1(\Sigma, \mathbb{Z}), \omega)$  that preserves the Hermitian structure and the flat connection. In this case the modular functor is defined by associating to  $\Sigma$ , the vector space  $Z^{(k)}(\Sigma)$  consisting of covariant constant sections of  $H^{(k)}$  over  $\mathcal{T}$ . So through the representation  $\rho$ , we get a representation  $\rho_k$  of the mapping class group  $\Gamma$  of  $\Sigma$  on  $Z^{(k)}(\Sigma)$ . In the  $\text{SU}(n)$ -case in the introduction this representation was denoted  $Z_k^{(n,d)}$ .

The 2+1 dimensional Chern–Simons theory also fits into a TQFT setup. Suppose  $Y$  is a compact oriented 3-manifold such that  $\partial Y = (-\Sigma_1) \cup \Sigma_2$ , where  $\Sigma_1$  and  $\Sigma_2$  are closed oriented surfaces and  $-\Sigma_1$  is  $\Sigma_1$  with reversed orientation. Assume furthermore that  $\gamma$  is a link inside  $Y - \partial Y$ . Then the TQFT-axioms states that there should be a linear morphism  $Z^{(k)}(Y, \gamma) : Z^{(k)}(\Sigma_1) \rightarrow Z^{(k)}(\Sigma_2)$ , which satisfies that gluing along boundary components goes to the corresponding composition of linear maps.

**Definition 10.** *The curve operator*

$$Z^{(k)}(Y, \gamma) : Z^{(k)}(\Sigma_1) \rightarrow Z^{(k)}(\Sigma_2),$$

*is defined to be*

$$Z^{(k)}(Y, \gamma) := T_{E_{I(Z)}(h_\gamma), I(Z)}^{(k)},$$

*where  $h_\gamma$  is the holonomy function associated to  $\gamma$ .*

To a simple closed curve  $\gamma$  on  $\Sigma$  the holonomy function  $h_\gamma \in C^\infty(M)$  is a pure phase function, i.e.  $h_\gamma = F_{r,s}$  where  $r, s \in \mathbb{Z}^n$ . Note that we do not label  $\gamma$  with an irreducible  $U(1)$ -representation  $\lambda$ .

Using this definition we could give the exact same proof as of Theorem 2 and obtain a classical theorem from the theory of theta functions.

**Theorem 14.** *Elements in the Torelli subgroup  $\ker \rho'$  are exactly those who are in the kernels of all  $\rho'_k$ s,*

$$\bigcap_{k=1}^{\infty} \ker \rho_k = \ker \rho'.$$

To this end we want to give a proof of Theorem 1 from [A9] in the case of abelian moduli spaces. We do this by studying the Hilbert–Schmidt norm of the curve operators.

**Definition 11.** *The Hilbert–Schmidt inner product of two operator  $A, B$  is*

$$\langle A, B \rangle = \text{Tr}(AB^*).$$

If we introduce the notation

$$\begin{aligned}\eta_k(r, s) &= \operatorname{Re}(e^{-\frac{\pi i}{k} r \cdot \bar{Z} r} e^{-2\pi i s \cdot \alpha} e^{-\pi^2 (s - \bar{Z} r) \cdot (2\pi k Y)^{-1} (s - \bar{Z} r)}) \\ &= e^{-\frac{\pi}{2k} ((s - Xr) \cdot Y^{-1} (s - Xr) + r \cdot Yr)}\end{aligned}$$

and recall the matrix coefficients of the Toeplitz operators  $T_{F_{r,s}}^{(k)}$  in terms of the basis of theta functions then

$$(F_{r,s} \theta_\alpha^{(k)}, \theta_\beta^{(k)})_Y = \delta_{\alpha - \beta, -[\frac{r}{k}]} e^{-2\pi i s \cdot \alpha} e^{-\frac{\pi i}{k} r \cdot s} \eta_k(r, s).$$

Note that  $f(r, s, Z)(k) = \eta_k(r, s)^{-1}$ . Here we suppress the  $Z$  dependence in  $\eta_k(r, s)$  since we from now only consider fixed Kähler structure.

**Lemma 4.**

$$\operatorname{Tr}(T_{F_{r,s}}^{(k)} (T_{F_{t,u}}^{(k)})^*) = \begin{cases} k^n \eta_k(r, s) \eta_k(t, u) \epsilon(r, s, t, u) & (r, s) \equiv (t, u) \pmod{k} \\ 0 & \text{else} \end{cases}$$

where  $\epsilon(r, s, t, u) \in \{\pm 1\}$  and is 1 for  $(r, s) = (t, u)$ .

*Proof.* We start by calculating the matrix coefficients of the product of the Toeplitz operators

$$\begin{aligned}(T_{F_{r,s}}^{(k)} (T_{F_{t,u}}^{(k)})^*)_{\beta\alpha} &= \sum_{\phi} (T_{F_{r,s}}^{(k)})_{\beta\phi} \overline{(T_{F_{t,u}}^{(k)})_{\alpha\phi}} \\ &= \delta_{\alpha - \beta, -[\frac{r-t}{k}]} e^{-2\pi i \alpha \cdot (s-u)} e^{-\frac{\pi i}{k} (r \cdot s - 2s \cdot t + t \cdot u)} \eta_k(r, s) \eta_k(t, u).\end{aligned}$$

Now when taking the trace  $\alpha = \beta$  and to get something non-zero we must have  $r \equiv t \pmod{k}$ . In that case

$$\operatorname{Tr}(T_{F_{r,s}}^{(k)} (T_{F_{t,u}}^{(k)})^*) = \epsilon(r, s, t, u) \eta_k(r, s) \eta_k(t, u) e^{\frac{\pi i}{k} r \cdot (s-u)} \sum_{\alpha} e^{-2\pi i \alpha (s-u)},$$

the  $\epsilon$  is obtained since  $t = r + kv$  only determines the equality

$$e^{-\frac{\pi i}{k} (r \cdot s - 2s \cdot t + t \cdot u)} = \pm e^{\frac{\pi i}{k} (r \cdot (s-u))}$$

up to a sign. Now if  $s \not\equiv u$  the last term is zero since it is  $n$  sums of all  $k$ 'th roots of unity, and hence 0. If  $s \equiv u$  each term in the sum is 1, and we get the desired result.  $\square$

Using the above lemma and the following limits

$$(12) \quad \lim_{k \rightarrow \infty} \eta_k(r, s) = 1 \quad \text{and} \quad \lim_{k \rightarrow \infty} \eta_k(r + kt, s + ku) = 0,$$

for all  $r, s \in \mathbb{Z}^n$ , we can prove the following

**Theorem 15.** *For any two smooth functions  $f, g \in C^\infty(M)$  and any  $Z \in \mathbb{H}$  one has that*

$$\langle f, g \rangle = \lim_{k \rightarrow \infty} k^{-n} \left\langle T_{f, I(Z)}^{(k)}, T_{g, I(Z)}^{(k)} \right\rangle,$$

where the real dimension of  $M$  is  $2n$ .

*Proof.* From Lemma 4 we get in particular

$$\|T_{F_{r,s}}^{(k)}\|_k = k^{-n/2} \sqrt{\operatorname{Tr}(T_{F_{r,s}}^{(k)} (T_{F_{r,s}}^{(k)})^*)} = \eta_k(r, s),$$

and

$$\|T_{E_I(F_{r,s})}^{(k)}\|_k = 1,$$

where  $\|\cdot\|_k = k^{-n/2} \sqrt{\langle \cdot, \cdot \rangle}$  is the  $k$ -scaled Hilbert–Schmidt norm.

Let  $f, g \in C^\infty(M)$  be an arbitrary elements and expand them in Fourier series

$$f = \sum_{(r,s) \in \mathbb{Z}^{2n}} \lambda_{r,s} F_{r,s} \quad \text{and} \quad g = \sum_{(t,u) \in \mathbb{Z}^{2n}} \mu_{t,u} F_{t,u}.$$

$\eta_k(r, s)$  and  $\eta_k(t, u)$  decays very fast for increasing  $r, s \in \mathbb{Z}^n$  and we have

$$\begin{aligned} k^{-n} \operatorname{Tr}(T_f^{(k)} (T_g^{(k)})^*) &= k^{-n} \sum_{(r,s), (t,u) \in \mathbb{Z}^{2n}} \lambda_{r,s} \bar{\mu}_{t,u} \operatorname{Tr}(T_{F_{r,s}}^{(k)} (T_{F_{t,u}}^{(k)})^*) \\ &= \sum_{(r,s) \in \mathbb{Z}^{2n}} \lambda_{r,s} \bar{\mu}_{t,u} \eta_k(r, s)^2 \\ &\quad + \sum_{\substack{(r,s), (t,u) \in \mathbb{Z}^{2n} \\ (t,u) \neq (0,0)}} \lambda_{r,s} \bar{\mu}_{r+kt, s+ku} \eta_k(r, s) \eta_k(r+kt, s+ku) \epsilon(r, s, t, u). \end{aligned}$$

This sum converges uniformly so if we take the large  $k$  limit we can interchange limit and summation. Now by Equation 12 and since

$$\lim_{k \rightarrow \infty} \mu_{r+kt, s+ku} = 0$$

by pointwise convergence of the Fourier series we finally get

$$\lim_{k \rightarrow \infty} k^{-n} \operatorname{Tr}(T_f^{(k)} (T_g^{(k)})^*) = \sum_{(r,s) \in \mathbb{Z}^{2n}} \lambda_{r,s} \bar{\mu}_{r,s}.$$

Now since the pure phase functions are orthogonal we get to desired result.  $\square$

It should be remarked that Theorem 15 just is a particular case of a theorem of the same wording, with  $M$  being a compact Kähler manifold, see e.g. [A9]. Theorem 15 was also proved in [BHSS] but only for a small class of principal polarized abelian varieties.

As a corollary to the proof of Theorem 15 we have

**Corollary 2.**

$$\langle f, g \rangle = \lim_{k \rightarrow \infty} k^{-n} \left\langle T_{E_I(f)}^{(k)}, T_{E_I(g)}^{(k)} \right\rangle.$$

We can interpret Corollary 2 in terms of TQFT curve operators. Since we defined a curve operator  $Z^{(k)}(\Sigma, \gamma)$  to be  $T_{E_I(h_\gamma)}^{(k)}$  where  $h_\gamma$  is the corresponding holonomy function of  $\gamma$  we immediately get

$$\langle h_{\gamma_1}, h_{\gamma_2} \rangle = \lim_{k \rightarrow \infty} k^{-n} \left\langle Z^{(k)}(\Sigma, \gamma_1), Z^{(k)}(\Sigma, \gamma_2) \right\rangle,$$

which was proved in [A9] and [MN].

Another interpretation is that gluing two cylinders  $(\Sigma \times [0, 1], \gamma_1)$  and  $(\Sigma \times [0, 1], \gamma_2)$  along  $\Sigma \times \{0\}$  and  $-\Sigma \times \{0\}$  and again at the top  $\Sigma \times \{1\}$  along  $-\Sigma \times \{1\}$ , we obtain the closed three manifold  $\Sigma \times S^1$  with the link  $\gamma_1 \cup \gamma_2^*$  embedded. Here  $\gamma_2^*$  means  $\gamma_2$  with reversed orientation. The TQFT gluing axioms now say that

$$Z^{(k)}(\Sigma \times S^1, \gamma_1 \cup \gamma_2^*) = \operatorname{Tr}(Z^{(k)}(\Sigma \times [0, 1], \gamma_1) Z^{(k)}(\Sigma \times [0, 1], \gamma_2^*)).$$

If we now define  $Z^{(k)}(\Sigma \times S^1, \gamma_1 \cup \gamma_2^*)$  to be exactly this, we see that if we take  $\gamma_1$  and  $\gamma_2$  to be the empty links we have

$$Z^{(k)}(\Sigma \times S^1) = \operatorname{Tr}(T_1^{(k)} (T_1^{(k)})^*) = k^n = \dim(Z^{(k)}(\Sigma)),$$

as it should be according to the axioms.



## 7. APPENDIX

In this appendix we provide the calculations needed to prove the explicit formulæ for the formal Hitchin connection given in Theorem 13.

We first observe that the theorem will follow from Equation 4 if we can show that for  $i \neq j$

$$\tilde{G}\left(\frac{\partial}{\partial Z_{ij}}\right) = 2i\frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_j} + 2i\frac{\partial}{\partial z_j} \otimes \frac{\partial}{\partial z_i} \quad \text{and} \quad \tilde{G}\left(\frac{\partial}{\partial \bar{Z}_{ij}}\right) = 2i\frac{\partial}{\partial \bar{z}_i} \otimes \frac{\partial}{\partial \bar{z}_j} + 2i\frac{\partial}{\partial \bar{z}_j} \otimes \frac{\partial}{\partial \bar{z}_i}$$

and for  $i = j$

$$\tilde{G}\left(\frac{\partial}{\partial Z_{ii}}\right) = 2i\frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_i} \quad \text{and} \quad \tilde{G}\left(\frac{\partial}{\partial \bar{Z}_{ii}}\right) = 2i\frac{\partial}{\partial \bar{z}_i} \otimes \frac{\partial}{\partial \bar{z}_i}.$$

Since the family of Kähler structures parametrized by  $\mathbb{H}$  is holomorphic we just have to solve the equations

$$G\left(\frac{\partial}{\partial Z_{ij}}\right) \cdot \omega = \frac{\partial I(Z)}{\partial Z_{ij}} \quad \text{and} \quad \bar{G}\left(\frac{\partial}{\partial \bar{Z}_{ij}}\right) \cdot \omega = \frac{\partial I(Z)}{\partial \bar{Z}_{ij}}.$$

**Lemma 5.** *The Kähler structure associated to a  $Z = X + iY \in \mathbb{H}$  is*

$$I(Z) = \begin{pmatrix} -Y^{-1}X & -(Y + XY^{-1}X) \\ Y^{-1} & XY^{-1} \end{pmatrix},$$

where we have written it as tensor in the frame  $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}$  of the tangent bundle  $TM$ .

*Proof.* This follows from the fact that the complex frame of  $TM$  are eigenvectors for  $I(Z)$ , that is

$$I(Z)\left(\frac{\partial}{\partial z_i}\right) = i\frac{\partial}{\partial z_i} \quad \text{and} \quad I(Z)\left(\frac{\partial}{\partial \bar{z}_i}\right) = -i\frac{\partial}{\partial \bar{z}_i},$$

and that

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{i}{2}Y^{-1}\bar{Z}\frac{\partial}{\partial x} + \frac{1}{2i}Y^{-1}\frac{\partial}{\partial y} = \frac{i}{2}(Y^{-1}X - i)\frac{\partial}{\partial x} + \frac{1}{2i}Y^{-1}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2i}Y^{-1}Z\frac{\partial}{\partial x} + \frac{i}{2}Y^{-1}\frac{\partial}{\partial y} = \frac{1}{2i}(Y^{-1}X + i)\frac{\partial}{\partial x} + \frac{i}{2}Y^{-1}\frac{\partial}{\partial y}. \end{aligned}$$

□

In the above we used the vector notation  $\frac{\partial}{\partial z}$  meaning an  $n$ -tuple of vectors  $\frac{\partial}{\partial z_i}$ . This convention eases the following calculations and will be used in the following.

*Proof of Theorem 13.* To this end we also need to recall the following derivation property for matrices. If  $A = (a_{ij})$  is a symmetric invertible  $n \times n$ -matrix then

$$\frac{\partial A^{-1}}{\partial a_{ij}} = -A^{-1} \frac{\partial A}{\partial a_{ij}} A^{-1} = -A^{-1} \Delta_{ij} A^{-1},$$

where  $\Delta_{ij}$  is an  $n \times n$ -matrix with all entries 0 except the  $ij$ 'th and  $ji$ 'th which is 1, if  $i \neq j$  and  $\Delta_{ii}$  is an  $n \times n$ -matrix with all entries 0 except the  $ii$ 'th diagonal entry which is 1. This follows easily from  $A^{-1}A = Id$ . Using this rule and that  $Y^{-1} = 2i(Z - \bar{Z})^{-1}$  we get

$$\frac{\partial Y^{-1}}{\partial Z_{ij}} = -\frac{1}{2i}Y^{-1}\Delta_{ij}Y^{-1}.$$

Derivation of the above equations with respect to  $Z_{ij}$  becomes rather messy if we do not also require  $Z$  to be normal, that is since  $Z$  is symmetric  $[Z, \bar{Z}] = 0$ , which is equivalent to  $[X, Y] = 0$ , or  $[X, Y^{-1}] = 0$ . A consequence of this is, that everything will commute even  $[Y^{-1}, \Delta_{ij}] = 0$  since the imaginary part of derivation of  $Z\bar{Z} = \bar{Z}Z$  with respect to  $Z_{ij}$  give  $Y\Delta_{ij} = \Delta_{ij}Y$ , and hence  $[Y^{-1}, \Delta_{ij}] = 0$ . Written as a tensor

$$\frac{\partial I(Z)}{\partial Z_{ij}} = \frac{1}{2i} Y^{-1} \Delta_{ij} Y^{-1} \begin{pmatrix} \bar{Z} & \bar{Z}^2 \\ -1 & -\bar{Z} \end{pmatrix}.$$

The symplectic form  $\omega = -\frac{1}{2i} \sum_{ij=1}^n w_{ij} dz_i \wedge d\bar{z}_j$  where  $Y^{-1} = W = (w_{ij})$ , should be contracted with  $G(\frac{\partial}{\partial Z_{ij}})$  we want to know its appearance in the  $Z$ -dependent  $\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$  frame. It is clear from above that

$$\frac{\partial I(Z)}{\partial Z_{ij}} \left( \frac{\partial}{\partial z} \right) = \frac{i}{2} Y^{-1} \bar{Z} \frac{\partial I(Z)}{\partial Z_{ij}} \left( \frac{\partial}{\partial x} \right) + \frac{1}{2i} Y^{-1} \frac{\partial I(Z)}{\partial Z_{ij}} \left( \frac{\partial}{\partial y} \right) = 0,$$

and an easy calculation shows that

$$\frac{\partial I(Z)}{\partial Z_{ij}} \left( \frac{\partial}{\partial \bar{z}} \right) = -Y^{-1} \Delta_{ij} \frac{\partial}{\partial \bar{z}}.$$

In other words

$$\frac{\partial I(Z)}{\partial Z_{ij}} = \begin{cases} -\sum_{k=1}^n (w_{ki} \frac{\partial}{\partial z_j} \otimes d\bar{z}_k + w_{kj} \frac{\partial}{\partial z_i} \otimes d\bar{z}_k) & \text{for } i \neq j \\ -\sum_{k=1}^n w_{ki} \frac{\partial}{\partial z_i} \otimes d\bar{z}_k & \text{for } i = j \end{cases}$$

Remark that since  $I(Z)^2 = -Id$ ,  $\frac{\partial I(Z)}{\partial Z_{ij}}$  and  $I(Z)$  anti-commute. This is clearly reflected in the above expressions for  $\frac{\partial I(Z)}{\partial Z_{ij}}$ . Now since  $G(\frac{\partial}{\partial Z_{ij}})$  is defined by

$$-G(\frac{\partial}{\partial Z_{ij}}) \cdot \frac{1}{2i} \sum_{kl=1}^n w_{kl} dz_k \wedge d\bar{z}_l = \frac{\partial I(Z)}{\partial Z_{ij}}$$

it is

$$G(\frac{\partial}{\partial Z_{ij}}) = \begin{cases} 2i \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_j} + 2i \frac{\partial}{\partial z_j} \otimes \frac{\partial}{\partial z_i} & \text{for } i \neq j \\ 2i \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial z_i} & \text{for } i = j. \end{cases}$$

With  $G(\frac{\partial}{\partial Z_{ij}})$  being expressed in complex coordinates, we should mentioned that the family of Kähler structures parametrized by  $\mathbb{H}$  in the way described above, actually is rigid, i.e.  $\bar{\partial}_Z(G(V)_Z) = 0$  for all vector field  $V$  on  $\mathbb{H}$ . This is clear since  $G(\frac{\partial}{\partial Z_{ij}})$  is zero in  $\bar{z}_i$  directions and  $G(\frac{\partial}{\partial Z_{ij}}) = 0$ .

We could do exactly the same thing with  $\frac{\partial}{\partial \bar{Z}_{ij}}$  and obtain

$$\frac{\partial I(Z)}{\partial \bar{Z}_{ij}} = -\frac{1}{2i} Y^{-1} \Delta_{ij} Y^{-1} \begin{pmatrix} Z & Z^2 \\ -1 & -Z \end{pmatrix}.$$

Again it is clear that

$$\frac{\partial I(Z)}{\partial \bar{Z}_{ij}} \left( \frac{\partial}{\partial z} \right) = 0 \quad \text{and} \quad \frac{\partial I(Z)}{\partial \bar{Z}_{ij}} \left( \frac{\partial}{\partial \bar{z}} \right) = -Y^{-1} \Delta_{ij} \frac{\partial}{\partial \bar{z}}.$$

In a similar way as above we obtain

$$\bar{G}(\frac{\partial}{\partial \bar{Z}_{ij}}) = \begin{cases} 2i \frac{\partial}{\partial \bar{z}_i} \otimes \frac{\partial}{\partial \bar{z}_j} + 2i \frac{\partial}{\partial \bar{z}_j} \otimes \frac{\partial}{\partial \bar{z}_i} & \text{for } i \neq j \\ 2i \frac{\partial}{\partial \bar{z}_i} \otimes \frac{\partial}{\partial \bar{z}_i} & \text{for } i = j. \end{cases} \quad \square$$

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